Some Limit Theorems for Linear Oscillators with Noise in the Coefficients*

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Abstract

Using the tools and methods developed in [1] limit theorems are proven for the linear oscillator with random coefficients. The asymptotic behaviour of the moments is studied in detail. The technique presented in this paper can be applied to general linear systems with noise and is well suited for the investigation of stochastic beam dynamics in accelerators.

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Contents

1 Linear Oscillator with Noise in the Coefficients

Starting point of our investigation is a nondegenerate ($\omega_0 \neq 0$) damped linear oscillator under the influence of noise

$$\ddot{x} + \varepsilon\,(\gamma(t) + \varepsilon\,\alpha)\,\dot{x} + \omega_0^2 \left(1 + \frac{\varepsilon}{\omega_0}\,\eta(t)\right) x = \varepsilon\,\omega_0\,\xi(t) \tag{1}$$

or written as a system of two first-order differential equations

$$\begin{cases} \dot{x} = \omega_0\, z \\ \\ \dot{z} = -\omega_0\, x + \varepsilon\,(\xi - \gamma\, z - \eta\, x) - \varepsilon^2\,\alpha\, z \end{cases} \tag{2}$$

ε is small parameter $|\varepsilon| < 1$.

The ε^2 proportionality of the deterministic term in the damping part is connected with the fact that we will discuss the dynamics on time scales $O(1/\varepsilon^2)$ (it is the minimum time scale where the stochastic effects could essentially influence the dynamics of our oscillator). If the damping will be weaker it will not affect the dynamics and we can neglect it, and if it will be stronger it will completely change the picture of the dynamics, the typical time scales become exponentially large $O(\exp(1/\varepsilon^a)), a > 0$ for positive damping and it will require other methods (see, for example [2]) that are beyond the scope of this paper.

Noise has been introduced in the damping part ($\gamma(t)$), as a modulation of the frequency ω_0 ($\eta(t)$) and as an external driving force $\xi(t)$.

As a model of noise we shall take stochastic processes defined by the following scalar products

$$\eta(t) = \vec{b}(t) \cdot \vec{y}(t), \qquad \xi(t) = \vec{h}(t) \cdot \vec{y}(t), \qquad \gamma(t) = \vec{d}(t) \cdot \vec{y}(t)$$

with nonrandom n-dimensional vectors $\vec{b}$, $\vec{h}$ and $\vec{d}$ which are quasiperiodic in t and which can be expanded into Fourier series

$$\vec{b}(t) = \sum_{m=-\infty}^{+\infty} \vec{b}_m \exp(i\nu_m t), \qquad \vec{b}_{-m} = \left(\vec{b}_m\right)^*$$

$$\vec{h}(t) = \sum_{m=-\infty}^{+\infty} \vec{h}_m \exp(i\nu_m t), \qquad \vec{h}_{-m} = \left(\vec{h}_m\right)^*$$

$$\vec{d}(t) \;=\; \sum_{m=-\infty}^{+\infty} \vec{d}_m \exp(i\nu_m t), \qquad \vec{d}_{-m} = \left(\vec{d}_m\right)^*$$

with real frequencies ν_m satisfying the condition

$$\nu_l \,+\, \nu_m \;=\; 0 \;\Leftrightarrow\; m \,+\, l \;=\; 0.$$

In the main part of this paper the vector $\vec{y}(t) \in R^n$ is assumed to be a solution of the linear system of Ito's stochastic differential equations

$$d\,\vec{y} \;=\; A\,\vec{y}\cdot dt \;+\; B\,d\vec{w}(t) \tag{3}$$

where A and B are $(n \times n)$ and $(n \times r)$ real constant matrices respectively, and $\vec{w}(t)$ is an r-dimensional Brownian motion, other choices for the noise model will be described later on.

As smoothness properties of the vector functions $\vec{b}$, $\vec{h}$ and $\vec{d}$ we shall require the convergence of the series [1]

$$\sum_{m=-\infty}^{+\infty} |\nu_m|^p \left(\left|\vec{b}_m\right| + \left|\vec{h}_m\right| + \left|\vec{d}_m\right|\right) \;<\; \infty, \qquad p \;=\; 0, 1 \tag{4}$$

We denote

$$F \;=\; \left\{\, p \,\in\, Z \;:\; |\vec{b}_p| + |\vec{h}_p| + |\vec{d}_p| \;\neq\; 0 \right\}$$

and introduce

$$F_k \;=\; \{(p,\, q) \;\in\; F \times F \;:\; |\nu_p + \nu_q - k\,\omega_0| \;\neq\; 0\}\,.$$

Besides the smoothness condition (4) we also require

$$\min_{k \,\in\, \{0,1,2,3,4\}} \;\; \inf_{(p,\,q) \,\in\, F_k} |\nu_p \;+\; \nu_q \;-\; k\,\omega_0| \;\geq\; \delta_f^2 \;>\; 0. \tag{5}$$

The condition (5) does not exclude resonances but requires them to be isolated. This can be easily changed to some kind of Diophantine conditions

[1] For a complex vector $\vec{w} \in C^n$ we use the usual spherical norm $|\vec{w}| = \sqrt{\vec{w} \cdot \vec{w}} = \sqrt{w_1 \cdot w_1^* + \ldots + w_n \cdot w_n^*}$ and for a $(n \times n)$ matrices with complex coefficients we shall use the norm $|M| = \sqrt{\lambda}$ where λ is the greatest eigenvalue of the matrix $M^* M$, which is compatible with the spherical norm for vectors.

with increasing smoothness properties (4). Note that (5) is always satisfied for periodic functions (i.e. $\nu_p = p \cdot \nu$) and for finite trigonometrical polynomials with arbitrary frequencies.

In this paper we will assume that all eigenvalues λ_k of the matrix A in (3) have negative real parts, i.e.

$$Re\,\lambda_k \;\leq\; -\delta_s^2 \;<\; 0, \qquad k = 1, \ldots, n \tag{6}$$

From this it follows (see, for example [3]) that if the initial random vector $\vec{y}_0$, independent of the r-dimensional Brownian motion $\vec{w}(t) - \vec{w}(0)$ for $0 < t < \infty$, has a normal distribution with mean value $\langle \vec{y}_0 \rangle = \vec{0}$ and covariance matrix

$$\left\langle \vec{y}_0 \cdot \vec{y}_0^\top \right\rangle \;=\; \int\limits_0^\infty \exp(\tau A)\, BB^\top \exp(\tau A^\top)\, d\tau \;\stackrel{\text{def}}{=}\; D$$

then the solution of (3) $\vec{y}(t, \vec{y}_0)$ is a stationary, zero-mean Gaussian process, with covariance function

$$\rho(\tau) \;=\; \begin{cases} \exp(\tau A)\, D & ; \quad \tau \geq 0 \\ \\ D\, \exp(\tau A^\top) & ; \quad \tau \leq 0 \end{cases} \tag{7}$$

Although, later on we shall not restrict the initial conditions for $\vec{y}$ in our noise model to be equal to the above mentioned initial conditions generating stationary solutions of the system (3) [2], all results will nevertheless be expressed in terms of the spectral density associated with the covariance function (7) $\Psi(\omega) = \Psi_c(\omega) - i\,\Psi_s(\omega)$ where [3]

$$\Psi_c(\omega) \;=\; \int\limits_0^\infty \cos(\omega\tau)\rho(\tau)d\tau \;=\; -\frac{A}{A^2 + \omega^2 I} \cdot D$$

[2]For simplicity we even shall take the initial condition to be a point in n-dimensional Euclidean space, but if one will follow the proofs of the theorems it will be clear that all results of this paper will be correct if we use as initial condition an arbitrary random vector, independent of the r-dimensional Brownian motion $\vec{w}(t) - \vec{w}(0)$ for $0 < t < \infty$, additionally assuming that some moments of $\vec{y}_0$ are finite.

[3]Note that if the matrices A and B^{-1} commute we use notation $\frac{A}{B}$ for the product AB^{-1}.

$$\Psi_s(\omega) \;=\; \int\limits_0^\infty \sin(\omega\tau)\rho(\tau)d\tau \;=\; \frac{\omega I}{A^2+\omega^2 I}\cdot D$$

For further purposes let us note that independently from real ω the norm of the matrix $\Psi(\omega)$ admits the estimate

$$|\Psi(\omega)| \;\leq\; \bar{C} \tag{8}$$

where $\bar{C}$ is some positive constant whose exact value depends on δ_s^2 and $\left|BB^\top\right|$ and is unimportant for us.

2 Special Basis in the Space of Polynomials

Often, the influence of noise in systems such as (1) is studied by considering its influence on the unperturbed invariants of motion such as energy

$$r = \frac{1}{2}\left(x^2+z^2\right)$$

or functions of the energy. For our later study of arbitrary moments we introduce a special time dependent (non-autonomous) basis in the space of polynomials.

For all nonnegative integers m, k we define

$$I_{m,\,k} \;=\; \exp\left(i\,(m-k)\,\omega_0\,t\right)\left(\frac{x+iz}{2}\right)^m\left(\frac{x-iz}{2}\right)^k$$

It is easy to check that the functions introduced above admit the following properties

a. $\left(\frac{\partial}{\partial t} \;+\; \omega_0\left(z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}\right)\right)\, I_{m,\,k} \;=\; 0$

b. $I_{m_1,\,k_1}\cdot I_{m_2,\,k_2} \;=\; I_{m_1+m_2,\,k_1+k_2}$

c. $I_{m,\,k} \;=\; I^*_{k,\,m}$

d. $I_{m,\,m} \;=\; \left(\frac{r}{2}\right)^m$

e. $|I_{m,\,k}|^2 \;=\; I_{m,\,k}\;\cdot\;I^*_{m,\,k} \;=\; \left(\frac{r}{2}\right)^{m+k}$

Representing x and z as

$$x = \frac{x+iz}{2} + \frac{x-iz}{2} = \exp(i\omega_0 t) I_{0,\,1} + \exp(-i\omega_0 t) I_{1,\,0}$$

$$z = \frac{x+iz}{2i} - \frac{x-iz}{2i} = i\exp(i\omega_0 t) I_{0,\,1} - i\exp(-i\omega_0 t) I_{1,\,0}$$

and using property **b** we can express $x^{m-k} \cdot z^k$ $(0 \le k \le m)$ with the help of the binomial theorem in the form of a linear combination of the functions $I_{p,\,q}$

$$x^{m-k} \cdot z^k = (i)^k \cdot$$

$$\cdot \sum_{p=0}^{m-k} \sum_{q=0}^{k} (-1)^q \binom{m-k}{p} \binom{k}{q} \exp(i(m - 2(p+q))\, \omega_0 t) \cdot I_{p+q,\, m-(p+q)}.$$

For $m \neq k$ $I_{m,\,k}$ are functions with complex values. However, we can also use as basis real valued functions $U_{m,\,k}$ and $V_{m,\,k}$ which are defined by

$$U_{m,\,k} = \frac{I_{m,\,k} + I_{k,\,m}}{2} = U_{k,\,m}, \qquad V_{m,\,k} = \frac{I_{m,\,k} - I_{k,\,m}}{2i} = -V_{k,\,m}.$$

Note further that the functions $U_{m,\,k}$ and $V_{m,\,k}$ can be easily expressed through the real valued functions $\bar{U}_{m,\,k}$ and $\bar{V}_{m,\,k}$

$$\bar{U}_{m,\,k} = \frac{\left(\frac{x+iz}{2}\right)^m \left(\frac{x-iz}{2}\right)^k + \left(\frac{x+iz}{2}\right)^k \left(\frac{x-iz}{2}\right)^m}{2}$$

$$\bar{V}_{m,\,k} = \frac{\left(\frac{x+iz}{2}\right)^m \left(\frac{x-iz}{2}\right)^k - \left(\frac{x+iz}{2}\right)^k \left(\frac{x-iz}{2}\right)^m}{2\,i}$$

which do not depend on time t with help of the following simple formula

$$\begin{pmatrix} U_{m,\,k} \\ V_{m,\,k} \end{pmatrix} = \begin{pmatrix} \cos((m-k)\,\omega_0 t) & -\sin((m-k)\,\omega_0 t) \\ \sin((m-k)\,\omega_0 t) & \cos((m-k)\,\omega_0 t) \end{pmatrix} \cdot \begin{pmatrix} \bar{U}_{m,\,k} \\ \bar{V}_{m,\,k} \end{pmatrix}$$

3 Stopped Process

Although a suitable choice for A and B in (3) allows one to approximate a wide range of spectral functions (with appropriate choice of A and B one can obtain for the y_1 component of the vector $\vec{y}$ every spectral function which is the ratio of two polynomials), the solution of this equation has the disadvantage that it also allows with positive probability arbitrary big excursions during finite fixed time intervals. In order to remove this effect and also to apply our proof technique we have to freeze and truncate the process.

Let $c(\varepsilon)$ be some positive function of ε defined on the set $\varepsilon \neq 0$. For every natural m and for every point $\vec{y}_0 \in R^n$ we introduce a random value

$$\tau_m^\varepsilon = \tau_m^\varepsilon(\vec{y}_0) = \inf\{t \geq 0 : \ (t, \vec{y}(t)) \notin [0, m) \times \{\vec{y} : \ |\vec{y}| < c(\varepsilon)\}\}$$

where $\vec{y}(t)$ is the solution of the system (3) which with probability one satisfies the initial condition $\vec{y}(0) = \vec{y}_0$. So with probability one for $m_1 \leq m_2$

$$0 \leq \tau_{m_1}^\varepsilon \leq \tau_{m_2}^\varepsilon.$$

Then with probability one there exists a limit (finite or infinite) when $m \to \infty$ of the sequence τ_m^ε which we will denote as

$$\tau^\varepsilon(\vec{y}_0) \stackrel{\text{def}}{=} \lim_{m \to \infty} \tau_m^\varepsilon(\vec{y}_0).$$

In other words $\tau^\varepsilon(\vec{y}_0)$ is the exit time from an open ball $|\vec{y}| < c(\varepsilon)$ for the solution of (3) starting with probability one from initial point $\vec{y}_0$. Note that if the matrix $BB^\top$ is nondegenerate then this exit time is finite with probability one.

The joint solution of the systems (2), (3) $(x(t), z(t), \vec{y}(t))$ is a Markovian diffusion process in $(n+2)$-dimensional Euclidean space. Let $s_t^\varepsilon = \min\{t, \tau^\varepsilon\}$. For the noise model (3) for reasons which we explained above we shall not study the moments of the stochastic process $(x(t), z(t), \vec{y}(t))$, but the moments of the stochastic process $(x(s_t^\varepsilon), z(s_t^\varepsilon), \vec{y}(s_t^\varepsilon))$ (stopped process). We shall use the time scale $O(\varepsilon^{-2})$ and the difference between t and s_t^ε for this time scale can be estimated with the help of the following

Theorem A: *There exist positive constants a and b so that for any initial point $\vec{y}_0$ and for any positive L*

$$P\left(\tau^\varepsilon < \frac{L}{\varepsilon^2}\right) \leq \left(\exp(a\,|\vec{y}_0|^2) + a\,\frac{L}{\varepsilon^2}\right)\exp(-b\,c^2(\varepsilon)) \tag{9}$$

Rewriting the left hand side of the inequality (9) in the form

$$P\left(\tau^{\varepsilon} < \frac{L}{\varepsilon^2}\right) \;=\; P\left(\max_{0\le t\le L/\varepsilon^2} |\, t \,-\, s_t^{\varepsilon}\,| \,>\, 0\right)$$

we see that on the time scale considered the measure of the set where $t \neq s_t^{\varepsilon}$ will go to zero as $\varepsilon \to 0$ if $c^2(\varepsilon) \to \infty$ faster then $b^{-1}\log(\varepsilon^{-2})$. On the other hand to apply the technique of our proof we require that

$$\lim_{\varepsilon\to 0} \varepsilon c^3(\varepsilon) = 0$$

so that we can not allow $c(\varepsilon)$ go to infinity too fast.

4 Asymptotic Behaviour of Moments

Let us introduce functions $\bar{c}_l(m,\, k)$ of integer arguments $m,\, k \,\geq\, 0$ with the help of

$$\bar{c}_1(m,\, k) \;=\; \frac{m}{4}\left(\sum_{\nu_l-\nu_p\,=\,\omega_0} \Big\{(m-1)\,\Psi^*(\omega_0+\nu_p)\,\vec{h}_p\cdot\left(\vec{b}_l + i\vec{d}_l\right) \;-\right.$$

$$-\;(k+1)\,\Psi^*(2\omega_0+\nu_p)\,\left(\vec{b}_p + i\vec{d}_p\right)\cdot\vec{h}_l \,-$$

$$-\;k\,\Psi^*(\nu_p)\,\left(\vec{b}_p + i\vec{d}_p\right)\cdot\vec{h}_l \;-\; k\,\Psi^*(\omega_0+\nu_p)\,\vec{h}_p\cdot\left(\vec{b}_l - i\vec{d}_l\right)\Big\} \;+$$

$$\left. +\sum_{\nu_l-\nu_p\,=\,-\omega_0} \Big\{m\Psi^{\top}(\nu_p)\,\left(\vec{b}_p^{\,*} - i\vec{d}_p^{\,*}\right)\cdot\vec{h}_l^{\,*} - k\Psi^{\top}(\omega_0+\nu_p)\,\vec{h}_p^{\,*}\cdot\left(\vec{b}_l^{\,*} - i\vec{d}_l^{\,*}\right)\Big\}\right)$$

$$\bar{c}_2(m,\, k) \;=\; -\frac{m(m-1)}{4}\sum_{\nu_l-\nu_p\,=\,2\omega_0} \Psi^*(\omega_0+\nu_p)\,\vec{h}_p\cdot\vec{h}_l$$

$$\bar{c}_3(m,\, k) \;=\; \frac{m(m-1)}{4}\sum_{\nu_l-\nu_p\,=\,3\omega_0} \Big\{\Psi^*(\omega_0+\nu_p)\,\vec{h}_p\cdot\left(\vec{b}_l - i\vec{d}_l\right) \;+$$

$$+ \Psi^*(2\omega_0 + \nu_p) \ \left(\vec{b}_p + i\vec{d}_p\right) \cdot \vec{h}_l \Big\}$$

$$\bar{c}_4(m,\ k) \ = \ -\frac{m(m-1)}{4} \sum_{\nu_l - \nu_p \,=\, 4\omega_0} \Psi^*(2\omega_0 + \nu_p) \ \left(\vec{b}_p + i\vec{d}_p\right) \cdot \left(\vec{b}_l - i\vec{d}_l\right)$$

$$\bar{c}_5(m,\ k) \ = \ \frac{m}{4} \left(\sum_{\nu_l - \nu_p \,=\, 2\omega_0} \Big\{ k\ \Psi^*(\nu_p) \ \left(\vec{b}_p + i\vec{d}_p\right) \cdot \left(\vec{b}_l - i\vec{d}_l\right) \ + \right.$$

$$+ \ (k+1)\ \Psi^*(2\omega_0 + \nu_p) \ \left(\vec{b}_p + i\vec{d}_p\right) \cdot \left(\vec{b}_l - i\vec{d}_l\right) \ -$$

$$- \ (m-1)\ \Psi^*(2\omega_0 + \nu_p) \ \left(\vec{b}_p + i\vec{d}_p\right) \cdot \left(\vec{b}_l + i\vec{d}_l\right) \Big\} \ -$$

$$\left. - \sum_{\nu_l - \nu_p \,=\, -2\omega_0} m\ \Psi^\top(\nu_p) \ \left(\vec{b}_p^{\,*} - i\vec{d}_p^{\,*}\right) \cdot \left(\vec{b}_l^{\,*} - i\vec{d}_l^{\,*}\right) \right)$$

$$\bar{c}_6(m,\ k) \ = \ \frac{m\,k}{4} \sum_{p=-\infty}^{\infty} \Big[\ \Psi(\omega_0 + \nu_p) \ + \ \Psi^*(\omega_0 + \nu_p) \Big] \ \vec{h}_p \cdot \vec{h}_p$$

$$\bar{c}_7(m,\ k) \ =$$

$$- \, \frac{m+k}{2}\ \alpha \ + \ \sum_{p=-\infty}^{\infty} \left\{ \frac{m\,k}{4} \ \Big[\ \Psi(\nu_p) \ + \ \Psi^*(\nu_p) \Big] \ \left(\vec{b}_p + i\vec{d}_p\right) \cdot \left(\vec{b}_p + i\vec{d}_p\right) \ - \right.$$

$$- \, \frac{m^2}{4} \Psi(\nu_p) \ \left(\vec{b}_p - i\vec{d}_p\right) \cdot \left(\vec{b}_p + i\vec{d}_p\right) \ - \ \frac{k^2}{4} \Psi^*(\nu_p) \ \left(\vec{b}_p + i\vec{d}_p\right) \cdot \left(\vec{b}_p - i\vec{d}_p\right) \ +$$

$$\left. + \left[\frac{(m+1)k}{4} \Psi(2\omega_0 + \nu_p) \ + \ \frac{m(k+1)}{4} \Psi^*(2\omega_0 + \nu_p) \right] \left(\vec{b}_p + i\vec{d}_p\right) \cdot \left(\vec{b}_p + i\vec{d}_p\right) \right\}$$

By using (4) and (8) it is not hard to show that $c_l(m,\ k)$ are correctly defined because the series converge absolutely for every fixed values of m and k.

Now in correspondence with an arbitrary two index array $a_{m,\,k}$ and nonnegative integer N

$$m,\, k \;\geq\; 0, \qquad m \,+\, k \;\leq\; N$$

we define a vector $\vec{\mathcal{V}}(a_{m,\,k};\, N)$ with $(N+1)(N+2)/2$ components with the help of the rule

$$\mathcal{V}_l(a_{m,\,k};\, N) \;=\; a_{m,\,k}, \qquad l \;=\; \frac{(m+k)(m+k+1)}{2} \,+\, k \,+\, 1$$

This ordering corresponds to the following ordering of the elements of the array $a_{m,\,k}$ (take by rows)

$$\begin{array}{lllll}
a_{0,\,0} & & & & \\
a_{1,\,0} & a_{0,\,1} & & & \\
a_{2,\,0} & a_{1,\,1} & a_{0,\,2} & & \\
\vdots & & & & \\
a_{N,\,0} & a_{N-1,\,1} & a_{N-2,\,2} & \dots & a_{0,\,N}
\end{array}$$

Consider now the system of ordinary differential equations with constant coefficients

$$\frac{d}{d\tau}\,\vec{\mathcal{V}}(a_{m,\,k};\, N) \;=\; \bar{\mathcal{K}}_N\,\vec{\mathcal{V}}(a_{m,\,k};\, N) \tag{10}$$

generated with the help of the rule

$$\begin{aligned}
\tfrac{d}{d\tau}\,a_{m,\,k} \;=\;& \bar{c}_2(m,\,k)\,a_{m-2,\,k} &+\;& \bar{c}_2^*(k,\,m)\,a_{m,\,k-2} &+ \\
& \bar{c}_1(m,\,k)\,a_{m-1,\,k} &+\;& \bar{c}_1^*(k,\,m)\,a_{m,\,k-1} &+ \\
& \bar{c}_3(m,\,k)\,a_{m-2,\,k+1} &+\;& \bar{c}_3^*(k,\,m)\,a_{m+1,\,k-2} &+ \\
& \bar{c}_4(m,\,k)\,a_{m-2,\,k+2} &+\;& \bar{c}_4^*(k,\,m)\,a_{m+2,\,k-2} &+ \\
& \bar{c}_5(m,\,k)\,a_{m-1,\,k+1} &+\;& \bar{c}_5^*(k,\,m)\,a_{m+1,\,k-1} &+ \\
& \bar{c}_6(m,\,k)\,a_{m-1,\,k-1} &+\;& \bar{c}_7(m,\,k)\,a_{m,\,k} &
\end{aligned} \tag{11}$$

where on the right hand side of (11) we take into account only terms with nonnegative indices.

Theorem B: *Let the function* $c(\varepsilon)$ *satisfy the condition*

$$\lim_{\varepsilon\to 0} \varepsilon c^3(\varepsilon) = 0$$

Then for arbitrary initial points $x_0,\ z_0,\ \vec{y}_0$ *, and for arbitrary nonnegative integer* N *and positive* L

$$\lim_{\varepsilon\to 0} \max_{0\le t\le L/\varepsilon^2} \left|\left\langle \bar{\mathcal{M}}_N^{-1}(\varepsilon^2 s_t^\varepsilon)\, \vec{\mathcal{V}}(I_{m,\,k}(s_t^\varepsilon);\, N) \ -\ \vec{\mathcal{V}}(I_{m,\,k}(0);\, N)\right\rangle\right| = 0$$

where the matrix $\bar{\mathcal{M}}_N(\tau)$ *is the fundamental matrix solution of the system of linear ordinary differential equations with constant coefficients (10).*

Remark 1: For further purposes it is important to note that the statement of the theorem B can also be written in the form

$$\lim_{\varepsilon\to 0} \max_{0\le t\le L/\varepsilon^2} \left|\left\langle \vec{a}_N(\varepsilon^2 s_t^\varepsilon)\ \cdot \vec{\mathcal{V}}(I_{m,\,k}(s_t^\varepsilon),\, N) \ -\ \vec{a}_N(0)\cdot \vec{\mathcal{V}}(I_{m,\,k}(0),\, N)\right\rangle\right| = 0$$

where $\vec{a}_N(\tau)$ is an arbitrary $(N+1)(N+2)/2$-dimensional vector satisfying

$$\frac{d\vec{a}_N}{d\tau} \ = \ -\bar{\mathcal{K}}_N^\top\, \vec{a}_N \tag{12}$$

Remark 2: For physical applications one can neglect the small difference between t and s_t^ε (see theorem A)and we have

$$\left\langle \vec{\mathcal{V}}(I_{m,\,k}(t),\, N)\right\rangle \ \approx\ \bar{\mathcal{M}}_N(\varepsilon^2 t)\ \vec{\mathcal{V}}(I_{m,\,k}(0),\, N)$$

5 Nonresonant Case

Let us now define what we mean by nonresonant.

Definition: *We shall say that there are no resonances of order* $m \ge 0$ *if for all integers p, q such that* $(p,\,q)\ \in\ F\times F$

$$m\,\omega_0 \ \ne\ \nu_p\ +\ \nu_q$$

Definition: *We shall say that there are no resonances up to order* $m\ge 0$ *if for all integers p, q such that* $(p,\,q)\ \in\ F\times F$

$$k\,\omega_0\ \ne\ \nu_p\ +\ \nu_q \quad \text{for} \quad k\ =\ 1,\ldots,m$$

In the nonresonant case only the values of $\bar{c}_6(m,\, k)$ and $\bar{c}_7(m,\, k)$ will be different from zero. Introduce for them special notations

$$\mathcal{A}_{m,\, k} \;=\; \bar{c}_7(m,\, k), \qquad \mathcal{C}_{m,\, k} \;=\; \bar{c}_6(m,\, k)$$

Note that $\mathcal{C}_{m,\, k}$ is a symmetrical function of its arguments, i.e. $\mathcal{C}_{m,\, k} = \mathcal{C}_{k,\, m}$ and it is also a real valued function i.e. $\mathcal{C}_{m,\, k} = \mathcal{C}^*_{m,\, k}$, and the function $\mathcal{A}_{m,\, k}$ satisfies $\mathcal{A}_{m,\, k} \;=\; \mathcal{A}^*_{k,\, m}$.

For the following let us also introduce special notations for the real and imaginary parts of $\mathcal{A}_{m,\, k}$

$$\bar{\mathcal{A}}_{m,\, k} \;=\; \frac{\mathcal{A}_{m,\, k} \;+\; \mathcal{A}_{k,\, m}}{2}, \qquad \bar{\mathcal{B}}_{m,\, k} \;=\; \frac{\mathcal{A}_{m,\, k} \;-\; \mathcal{A}_{k,\, m}}{2i}$$

We shall call $\bar{\mathcal{A}}_{m,\, k}$ and $\bar{\mathcal{B}}_{m,\, k}$ for reasons which will become clear later diffusion coefficient and tune shift respectively. Note that $\bar{\mathcal{A}}_{m,\, m} \;=\; \mathcal{A}_{m,\, m}$ and $\bar{\mathcal{B}}_{m,\, m} \;=\; 0$.

Theorem C: *Let there be no resonances up to order* 4 *and let the function* $c(\varepsilon)$ *satisfy the condition*

$$\lim_{\varepsilon \to 0} \varepsilon c^3(\varepsilon) = 0$$

Then for any initial points $x_0,\; z_0,\; \vec{y}_0$ *, for any nonnegative integers* m *,* k *and for any positive* L

$$\lim_{\varepsilon \to 0} \quad \max_{0 \le t \le L/\varepsilon^2} \quad \left| \left\langle \sum_{p=0}^{q} a_p^{m,\, k}(\varepsilon^2 s_t^\varepsilon) I_{m-p,\, k-p}(s_t^\varepsilon) \;-\; \sum_{p=0}^{q} a_p^{m,\, k}(0) I_{m-p,\, k-p}(0) \right\rangle \right| \;=\; 0$$

where $q \;=\; \min\{m,\, k\}$ *and the functions* $a_p^{m,\, k}(\tau)$ *are an arbitrary solution of the system of linear ordinary differential equations with constant coefficients*

$$\frac{da_0^{m,\, k}}{d\tau} \;=\; -\mathcal{A}_{m,\, k}\, a_0^{m,\, k}$$

$$\frac{da_p^{m,\, k}}{d\tau} \;=\; -\mathcal{A}_{m-p,\, k-p}\, a_p^{m,\, k} \;-\; \mathcal{C}_{m-p+1,\, k-p+1}\, a_{p-1}^{m,\, k}$$

$$p \;=\; 1, \ldots, q$$

The proof of this theorem can be obtained from the remark to the theorem B with help of some straightforward calculations.

Remark 1: We would like to note that for the study of the behaviour of first order moments (i.e. when $m + k = 1$) we actually need to avoid resonances in theorem B up to order 2 only.

Remark 2: The general solution of the system of differential equations for the coefficients $a_p^{m,\,k}$ has the form

$$a_0^{m,\,k}(\tau) \;=\; a_0^{m,\,k}(0) \cdot \exp\left(-\mathcal{A}_{m,\,k}\,\tau\right)$$

$$a_p^{m,\,k}(\tau) \;=\; \left(a_p^{m,\,k}(0) - \mathcal{C}_{m-p+1,\,k-p+1} \int\limits_0^\tau a_{p-1}^{m,\,k}(\zeta) \cdot \exp\left(\mathcal{A}_{m-p,\,k-p}\,\zeta\right) d\zeta \right) \cdot$$

$$\cdot \exp\left(-\mathcal{A}_{m-p,\,k-p}\,\tau\right)$$

$$p \;=\; 1,\,\ldots,\,q$$

Choosing the initial conditions

$$a_0^{m,\,k}(0) \;=\; 1, \qquad a_p^{m,\,k}(0) \;=\; 0, \qquad p \;=\; 1,\,\ldots,\,q$$

the statement of the theorem C can be rewritten in the form

$$\lim_{\varepsilon\to 0} \quad \max_{0\le t\le L/\varepsilon^2}$$

$$\left| \left\langle \exp(-\varepsilon^2 \mathcal{A}_{m,\,k} s_t^\varepsilon) I_{m,\,k}(s_t^\varepsilon) - \left(I_{m,\,k}(0) - \sum_{p=1}^{q} a_p^{m,\,k}(\varepsilon^2 s_t^\varepsilon) I_{m-p,\,k-p}(s_t^\varepsilon) \right) \right\rangle \right| = 0$$

In the case when $m \neq k$ we can use the real valued functions $U_{m,\,k}$ and $V_{m,\,k}$ instead of the complex valued $I_{m,\,k}$. Due to the symmetries $U_{m,\,k} = U_{k,\,m}$ and $V_{m,\,k} = -V_{k,\,m}$ it is enough to consider only the case when $m > k$. So we have

Corollary 1: *Let there be no resonances up to order* 4 *and let the function* $c(\varepsilon)$ *satisfy the condition*

$$\lim_{\varepsilon\to 0} \varepsilon c^3(\varepsilon) = 0$$

Then for any initial points $x_0,\ z_0,\ \vec{y}_0$ *, for any nonnegative integers* $m,\ k$ *satisfying* $m\ >\ k$ *, and for any positive* L

$$\lim_{\varepsilon\to 0}\ \max_{0\le t\le L/\varepsilon^2}\ \left|\left\langle \sum_{p=0}^{k} M_p^{m,\,k}(\varepsilon^2 s_t^\varepsilon)\cdot \vec{W}_p^{m,\,k}(s_t^\varepsilon) - \sum_{p=0}^{k} M_p^{m,\,k}(0)\cdot \vec{W}_p^{m,\,k}(0)\right\rangle\right| = 0$$

where

$$\vec{W}_p^{m,\,k}(\tau) = \begin{pmatrix} U_{m-p,\,k-p}(\tau) \\ \\ V_{m-p,\,k-p}(\tau)\end{pmatrix}, \qquad M_p^{m,\,k}(\tau) = \begin{pmatrix} \alpha_p^{m,\,k}(\tau) & -\beta_p^{m,\,k}(\tau) \\ \\ \beta_p^{m,\,k}(\tau) & \alpha_p^{m,\,k}(\tau)\end{pmatrix}$$

and the functions $\alpha_p^{m,\,k}(\tau)$ *and* $\beta_p^{m,\,k}(\tau)$ *are an arbitrary real solution of the system of linear ordinary differential equations with constant coefficients*

$$\frac{d}{d\tau}\begin{pmatrix}\alpha_0^{m,\,k} \\ \\ \beta_0^{m,\,k}\end{pmatrix} = \bar{R}_0^{m,\,k}\begin{pmatrix}\alpha_0^{m,\,k} \\ \\ \beta_0^{m,\,k}\end{pmatrix}$$

$$\frac{d}{d\tau}\begin{pmatrix}\alpha_p^{m,\,k} \\ \\ \beta_p^{m,\,k}\end{pmatrix} = \bar{R}_p^{m,\,k}\begin{pmatrix}\alpha_p^{m,\,k} \\ \\ \beta_p^{m,\,k}\end{pmatrix} - \mathcal{C}_{m-p+1,\,k-p+1}\begin{pmatrix}\alpha_{p-1}^{m,\,k} \\ \\ \beta_{p-1}^{m,\,k}\end{pmatrix}$$

$$p\ =\ 1,\ldots,k$$

$$\bar{R}_p^{m,\,k} = \begin{pmatrix} -\bar{\mathcal{A}}^{m-p,\,k-p} & \bar{\mathcal{B}}^{m-p,\,k-p} \\ \\ -\bar{\mathcal{B}}^{m-p,\,k-p} & -\bar{\mathcal{A}}^{m-p,\,k-p}\end{pmatrix}$$

For the important particular case when we do not have an external noise in our system, i.e. $\xi(t)\ \equiv\ 0$ (that means that we can put $\vec{h}(t)\ \equiv\ \vec{0}$ and hence all $\mathcal{C}_{m,\,k}\ =\ 0$) the differential equations defining the functions $a_p^{m,\,m},\ \alpha_p^{m,\,k}$ and $\beta_p^{m,\,k}$ admit the simple solution

$$a_0^{m,\,m}(\tau)\ =\ \exp(-\bar{\mathcal{A}}_{m,\,m}\tau)\, a_0^{m,\,m}(0)$$

$$\begin{pmatrix}\alpha_0^{m,\,k}(\tau) \\ \\ \beta_0^{m,\,k}(\tau)\end{pmatrix} = \exp(-\bar{\mathcal{A}}_{m,\,k}\tau)\begin{pmatrix}\cos(\bar{\mathcal{B}}_{m,\,k}\tau) & \sin(\bar{\mathcal{B}}_{m,\,k}\tau) \\ \\ -\sin(\bar{\mathcal{B}}_{m,\,k}\tau) & \cos(\bar{\mathcal{B}}_{m,\,k}\tau)\end{pmatrix}\begin{pmatrix}\alpha_0^{m,\,k}(0) \\ \\ \beta_0^{m,\,k}(0)\end{pmatrix}$$

$$a_p^{m,\,m}(\tau)\;\equiv\;0,\qquad \alpha_p^{m,\,k}(\tau)\;\equiv\;0,\qquad \beta_p^{m,\,k}(\tau)\;\equiv\;0,\qquad p\neq 0$$

Choosing initial conditions

$$a_0^{m,\,m}(0)\;=\;1,\qquad \alpha_0^{m,\,k}(0)\;=\;1,\qquad \beta_0^{m,\,k}(0)\;=\;0$$

we get the following

Corollary 2: *Let* $\xi(t)\;\equiv\;0$ *and let there be no resonances of orders* 2 *and* 4, *and let the function* $c(\varepsilon)$ *satisfy the condition*

$$\lim_{\varepsilon\to 0}\varepsilon c^3(\varepsilon)=0$$

Then for any initial points $x_0,\;z_0,\;\vec{y}_0$, *for any nonnegative integers* $m,\;k$ *satisfying* $m\;\geq\;k$, *and for any positive* L

$$\lim_{\varepsilon\to 0}\;\max_{0\leq t\leq L/\varepsilon^2}\;\left|\left\langle \exp\left(-\varepsilon^2\,\bar{\mathcal{A}}_{m,\,m}\,s_t^\varepsilon\right) r^m\left(s_t^\varepsilon\right)-r^m(0)\right\rangle\right|=0$$

for $k\;=\;m$ *and*

$$\lim_{\varepsilon\to 0}\max_{0\leq t\leq L/\varepsilon^2}\left|\left\langle \exp\left(-\varepsilon^2\,\bar{\mathcal{A}}_{m,k}\,s_t^\varepsilon\right)\bar{M}_m^k(s_t^\varepsilon)\begin{pmatrix}\bar{U}_{m,k}(s_t^\varepsilon)\\ \\ \bar{V}_{m,k}(s_t^\varepsilon)\end{pmatrix}-\begin{pmatrix}\bar{U}_{m,k}(0)\\ \\ \bar{V}_{m,k}(0)\end{pmatrix}\right\rangle\right|=0$$

where

$$\bar{M}_m^k(\tau)=\begin{pmatrix}\cos\left(\Delta_{m,k}^\varepsilon\tau\right) & -\sin\left(\Delta_{m,k}^\varepsilon\tau\right)\\ \\ \sin\left(\Delta_{m,k}^\varepsilon\tau\right) & \cos\left(\Delta_{m,k}^\varepsilon\tau\right)\end{pmatrix},\quad \Delta_{m,k}^\varepsilon=(m-k)\,\omega_0-\varepsilon^2\bar{\mathcal{B}}_{m,k}$$

otherwise.

For the important case of constant vectors $\vec{b}$, $\vec{h}$ and $\vec{d}$ the formulae for $\bar{\mathcal{A}}_{m,\,k},\;\bar{\mathcal{B}}_{m,\,k}$ and $\mathcal{C}_{m,\,k}$ take the simplified form

$$\bar{\mathcal{A}}_{m,\,k}\;=\;-\frac{m+k}{2}\,\alpha\;-\;\frac{(m-k)^2}{4}\Psi_c(0)\,\vec{b}\cdot\vec{b}\;+\;\frac{(m+k)^2}{4}\Psi_c(0)\,\vec{d}\cdot\vec{d}\,+$$

$$+\frac{m+2mk+k}{4}\left[\Psi_c(2\omega_0)\,\vec{b}\cdot\vec{b}+\Psi_c(2\omega_0)\,\vec{d}\cdot\vec{d}+\left(\Psi_s(2\omega_0)-\Psi_s^\top(2\omega_0)\right)\,\vec{d}\cdot\vec{b}\right]$$

$$\bar{\mathcal{B}}_{m,\,k} \;=\; \frac{m^2-k^2}{4}\left(\Psi_c(0)\;+\;\Psi_c^\top(0)\right)\vec{d}\cdot\vec{b}+$$

$$+\;\frac{m-k}{4}\left[\Psi_s(2\omega_0)\,\vec{b}\cdot\vec{b}\;+\;\Psi_s(2\omega_0)\,\vec{d}\cdot\vec{d}\;-\;\left(\Psi_c(2\omega_0)\;-\;\Psi_c^\top(2\omega_0)\right)\vec{d}\cdot\vec{b}\right]$$

$$\mathcal{C}_{m,\,k} \;=\; \frac{mk}{2}\,\Psi_c(\omega_0)\,\vec{h}\cdot\vec{h}$$

6 First and Second Order Moments

First order moments in the nonresonant case:

Corollary C1: *Let there be no resonances up to order* 2 *and let the function* $c(\varepsilon)$ *satisfy the condition*

$$\lim_{\varepsilon\to 0}\varepsilon c^3(\varepsilon)=0$$

Then for any initial points $x_0,\; z_0,\; \vec{y}_0$ *and for any positive* L

$$\lim_{\varepsilon\to 0}\quad\max_{0\le t\le L/\varepsilon^2}\quad\left|\left\langle \exp\left(-\varepsilon^2\bar{\mathcal{A}}_{1,\,0}\,s_t^\varepsilon\right)\,M(s_t^\varepsilon)\begin{pmatrix} x(s_t^\varepsilon)\\ \\ z(s_t^\varepsilon)\end{pmatrix}-\begin{pmatrix} x_0\\ \\ z_0\end{pmatrix}\right\rangle\right|=0$$

where

$$M(\tau)=\begin{pmatrix}\cos\left(\left(\omega_0\;-\;\varepsilon^2\bar{\mathcal{B}}_{1,\,0}\right)\tau\right) & -\sin\left(\left(\omega_0\;-\;\varepsilon^2\bar{\mathcal{B}}_{1,\,0}\right)\tau\right)\\ \\ \sin\left(\left(\omega_0\;-\;\varepsilon^2\bar{\mathcal{B}}_{1,\,0}\right)\tau\right) & \cos\left(\left(\omega_0\;-\;\varepsilon^2\bar{\mathcal{B}}_{1,\,0}\right)\tau\right)\end{pmatrix}$$

Second order moments in nonresonant case:

Corollary C2: *Let there be no resonances up to order* 4 *and let the function* $c(\varepsilon)$ *satisfy the condition*

$$\lim_{\varepsilon\to 0}\varepsilon c^3(\varepsilon)=0.$$

Then for any initial points x_0, z_0, $\vec{y}_0\in R^n$ *and for any positive* L

$$\lim_{\varepsilon\to 0}\ \max_{0\le t\le L/\varepsilon^2}\ \left|\left\langle r\left(s_t^\varepsilon\right) - r_0 - \varepsilon^2\, 2\,\mathcal{C}_{1,\,1} s_t^\varepsilon\right\rangle\right| = 0$$

for $\bar{\mathcal{A}}_{1,\,1} = 0$ *and*

$$\lim_{\varepsilon\to 0}\ \max_{0\le t\le L/\varepsilon^2}\ \left|\left\langle\left(r\left(s_t^\varepsilon\right) + \frac{2\,\mathcal{C}_{1,\,1}}{\bar{\mathcal{A}}_{1,\,1}}\right)\exp\left(-\varepsilon^2\bar{\mathcal{A}}_{1,\,1} s_t^\varepsilon\right) - \left(r_0 + \frac{2\,\mathcal{C}_{1,\,1}}{\bar{\mathcal{A}}_{1,\,1}}\right)\right\rangle\right| = 0$$

otherwise.

To estimate the behaviour of the remainder of the second moments we shall use the functions

$$\bar{U}_{2,\,0} = \frac{x^2 - z^2}{4} \qquad \bar{V}_{2,\,0} = \frac{xz}{2}$$

Corollary C3: *Let there be no resonances up to order* 4 *and let the function* $c(\varepsilon)$ *satisfy the condition*

$$\lim_{\varepsilon\to 0}\varepsilon c^3(\varepsilon) = 0$$

Then for any initial points x_0, z_0, $\vec{y}_0$ *and for any positive* L

$$\lim_{\varepsilon\to 0}\ \max_{0\le t\le L/\varepsilon^2}\left|\left\langle\exp\left(-\varepsilon^2\,\bar{\mathcal{A}}_{2,\,0}\, s_t^\varepsilon\right)\ M(s_t^\varepsilon)\begin{pmatrix}\bar{U}_{2,\,0}(s_t^\varepsilon)\\ \\ \bar{V}_{2,\,0}(s_t^\varepsilon)\end{pmatrix} - \begin{pmatrix}\bar{U}_{2,\,0}(0)\\ \\ \bar{V}_{2,\,0}(0)\end{pmatrix}\right\rangle\right| = 0$$

where

$$M(\tau) = \begin{pmatrix}\cos\left(\left(2\omega_0 - \varepsilon^2\bar{\mathcal{B}}_{2,\,0}\right)\tau\right) & -\sin\left(\left(2\omega_0 - \varepsilon^2\bar{\mathcal{B}}_{2,\,0}\right)\tau\right)\\ \\ \sin\left(\left(2\omega_0 - \varepsilon^2\bar{\mathcal{B}}_{2,\,0}\right)\tau\right) & \cos\left(\left(2\omega_0 - \varepsilon^2\bar{\mathcal{B}}_{2,\,0}\right)\tau\right)\end{pmatrix}$$

7 Comparison with White Noise Model

As a special case we consider white noise in this chapter i.e.

$$\vec{y} = C\,\dot{\vec{w}}(t) \tag{13}$$

where C is a real constant $(n \times r)$ matrix and $\vec{w}(t)$ is an r-dimensional Brownian motion. Substituting (13) into (2) we have

$$\left\{\begin{array}{rcl} dx &=& \omega_0\, z\, dt \\ \\ dz &=& -\omega_0\, x\, dt \;-\; \varepsilon^2\, \alpha\, z\, dt \;+\; \varepsilon\, C^\top \left(\vec{h} - z\vec{d} - x\vec{b}\right) \cdot d\vec{w}(t) \end{array}\right. \tag{14}$$

As usual for the case of multiplicative noise we shall treat the system (14) as a system of Stratonovich's stochastic differential equations.

Introduce the matrix $\Phi = \frac{1}{2} C C^\top$ which plays the role of the spectral density for the noise model (13) and define functions $\breve{c}_l(m,\, k)$ with the help of

$$\breve{c}_1(m,\, k) \;=\; \frac{m}{2} \sum_{\nu_l - \nu_p \,=\, \omega_0} \left\{(m - 2k - 1)\, \Phi\, \vec{h}_p \cdot \vec{b}_l \;-\; i\,(m + 2k)\, \Phi\, \vec{h}_p \cdot \vec{d}_l\right\},$$

$$\breve{c}_2(m,\, k) \;=\; -\frac{m(m-1)}{4} \sum_{\nu_l - \nu_p \,=\, 2\omega_0} \Phi\, \vec{h}_p \cdot \vec{h}_l,$$

$$\breve{c}_3(m,\, k) \;=\; \frac{m(m-1)}{2} \sum_{\nu_l - \nu_p \,=\, 3\omega_0} \Phi\, \vec{h}_p \cdot \left(\vec{b}_l - i\vec{d}_l\right),$$

$$\breve{c}_4(m,\, k) \;=\; -\frac{m(m-1)}{4} \sum_{\nu_l - \nu_p \,=\, 4\omega_0} \Phi\, \left(\vec{b}_p + i\vec{d}_p\right) \cdot \left(\vec{b}_l - i\vec{d}_l\right),$$

$$\breve{c}_5(m,\, k) \;=\; \frac{m}{2} \sum_{\nu_l - \nu_p \,=\, 2\omega_0} \Big\{ -\,(m + k)\, \Phi\, \vec{d}_p \cdot \vec{d}_l \;+$$

$$+\; (k - m + 1)\, \Phi\, \vec{b}_p \cdot \vec{b}_l \;+\; i\,(2k + 1)\, \Phi\, \vec{b}_p \cdot \vec{d}_l \Big\},$$

$$\breve{c}_6(m,\ k)\ =\ \frac{mk}{2}\sum_{p\,=\,-\infty}^{\infty}\Phi\,\vec{h}_p\cdot\vec{h}_p,$$

$$\breve{c}_7(m,\ k)\ =\ -\frac{m+k}{2}\,\alpha\ +\ \frac{4mk-m(m-1)-k(k-1)}{4}\sum_{p\,=\,-\infty}^{\infty}\Phi\,\vec{b}_p\cdot\vec{b}_p\ +$$

$$+\ \frac{4mk+m(m+1)+k(k+1)}{4}\sum_{p\,=\,-\infty}^{\infty}\Phi\,\vec{d}_p\cdot\vec{d}_p\ +\ i\,\frac{m^2-k^2}{2}\sum_{p\,=\,-\infty}^{\infty}\Phi\,\vec{d}_p\cdot\vec{b}_p.$$

Consider now the system of ordinary differential equations with constant coefficients

$$\frac{d}{d\tau}\,\vec{\mathcal{V}}(a_{m,\,k};\ N)\ =\ \breve{\mathcal{K}}_N\,\vec{\mathcal{V}}(a_{m,\,k};\ N) \tag{15}$$

generated with the help of the rule

$$\begin{array}{rcllllll}
\frac{d}{d\tau}\,a_{m,\,k} & = & \breve{c}_2(m,\ k)\,a_{m-2,\,k} & + & \breve{c}_2^*(k,\ m)\,a_{m,\,k-2} & + \\
 & & \breve{c}_1(m,\ k)\,a_{m-1,\,k} & + & \breve{c}_1^*(k,\ m)\,a_{m,\,k-1} & + \\
 & & \breve{c}_3(m,\ k)\,a_{m-2,\,k+1} & + & \breve{c}_3^*(k,\ m)\,a_{m+1,\,k-2} & + \\
 & & \breve{c}_4(m,\ k)\,a_{m-2,\,k+2} & + & \breve{c}_4^*(k,\ m)\,a_{m+2,\,k-2} & + \\
 & & \breve{c}_5(m,\ k)\,a_{m-1,\,k+1} & + & \breve{c}_5^*(k,\ m)\,a_{m+1,\,k-1} & + \\
 & & \breve{c}_6(m,\ k)\,a_{m-1,\,k-1} & + & \breve{c}_7(m,\ k)\,a_{m,\,k} &
\end{array} \tag{16}$$

where on the right hand side of (16) we take into account only terms with nonnegative indices.

Theorem D: *For any initial points* $x_0,\ \ z_0,\ \ \vec{y}_0$ *, for any nonnegative integer* N *and for any positive* L

$$\lim_{\varepsilon\to 0}\ \max_{0\le t\le L/\varepsilon^2}\ \left|\left\langle\breve{\mathcal{M}}_N^{-1}(\varepsilon^2 t)\,\vec{\mathcal{V}}(I_{m,\,k}(t);\ N)\ -\ \vec{\mathcal{V}}(I_{m,\,k}(0);\ N)\right\rangle\right| = 0$$

where the matrix $\breve{\mathcal{M}}_N(\tau)$ is the fundamental matrix solution of the system of linear ordinary differential equations with constant coefficients (15).

Note that in this case and for the noise model introduced below we have not to distinguish between s_t^ε and t. We also mention that if we substitute into the expressions of $\bar{c}_l(m,\, k)$ the matrix Φ instead of the matrix Ψ ("spectral density" of white noise) we exactly get $\breve{c}_l(m,\, k)$.

8 Another Noise Model

The technique derived in this paper can be applied to a wide class of noise models. As a model of noise in this section we consider the stochastic processes represented by the following trigonometrical polynomials[4] (cosine and sine functions with random phases)

$$\eta(t) \;=\; \sum_{m=-q}^{q} \eta_m \exp\left(i\left(\nu_m t \;+\; \vec{v}_m \cdot \vec{y}\,\right)\right), \qquad \eta_{-m} = (\eta_m)^*$$

$$\xi(t) \;=\; \sum_{m=-q}^{q} \xi_m \exp\left(i\left(\nu_m t \;+\; \vec{v}_m \cdot \vec{y}\,\right)\right), \qquad \xi_{-m} = (\xi_m)^*$$

$$\gamma(t) \;=\; \sum_{m=-q}^{q} \gamma_m \exp\left(i\left(\nu_m t \;+\; \vec{v}_m \cdot \vec{y}\,\right)\right), \qquad \gamma_{-m} = (\gamma_m)^*$$

with real ν_m and $\vec{v}_m \,\in\, R^n$ satisfying the conditions

$$|\nu_l \;+\; \nu_m| \;+\; |\vec{v}_m \;+\; \vec{v}_l| \;=\; 0 \;\Leftrightarrow\; m \;+\; l \;=\; 0$$

where the integers $m,\, l$ obey $m,\, l \;=\; -q,\, \ldots,\, q.$

The vector $\vec{y} \,\in\, R^n$ is assumed to be a solution of the following Ito's system

$$d\,\vec{y} \;=\; \sqrt{2}\, B\, d\vec{w}(t)$$

[4]In order not to deal with conditions similar to (4) and (5) we consider the case of a finite trigonometrical sum. The extension to the case of infinite series and also the proof of the theorem E we leave as an exercise for the interested reader.

where B is a real constant $(n \times r)$ matrix and $\vec{w}(t)$ is an r-dimensional Brownian motion. For simplicity we assume that the $(n \times n)$ matrix $B\,B^\top$ is nondegenerate and $|\,\vec{v}_m\,| \neq 0$ for all $m = -q, \ldots, q$ (i.e. we do not have deterministic harmonics in our perturbation model).

For $p = -q, \ldots, q$ we introduce real vectors $\vec{u}_p = B^\top \vec{v}_p \in R^r$ which satisfy $|\vec{u}_p| \neq 0$, and a function $\Omega\,(\omega,\, \vec{u}_p)$

$$\Omega\,(\omega,\, \vec{u}_p) \;=\; \frac{|\vec{u}_p|^2 \;+\; i\,\omega}{|\vec{u}_p|^4 \;\;+\;\; \omega^2}$$

and define $\tilde{c}_l(m,\, k)$ as follow

$$\tilde{c}_1(m,\, k) \;=\; \frac{m}{4}\Bigg(\sum_{\substack{|\nu_p+\nu_l+\omega_0|+\\+|\vec{v}_p+\vec{v}_l|=0}} \{(m-1)\,\Omega\,(\omega_0 \,+\, \nu_p,\, \vec{u}_p)\;\xi_p\;(\eta_l - i\gamma_l)\;-$$

$$-\;(k+1)\,\Omega\,(2\omega_0 \,+\, \nu_p,\, \vec{u}_p)\;(\eta_p + i\,\gamma_p)\;\xi_l\;-$$

$$-\;k\,\Omega\,(\omega_0 \,+\, \nu_p,\, \vec{u}_p)\;\xi_p\;(\eta_l + i\,\gamma_l)\;-\;k\,\Omega\,(\nu_p,\, \vec{u}_p)\;(\eta_p + i\,\gamma_p)\;\xi_l\}\;+$$

$$+\sum_{\substack{|\nu_l-\nu_p+\omega_0|+\\+|\vec{v}_l-\vec{v}_p|=0}} \Big\{m\,\Omega^*\,(\nu_p,\, \vec{u}_p)\;\Big(\eta_p^* - i\gamma_p^*\Big)\;\xi_l \;-\; k\,\Omega^*\,(\omega_0 \,+\, \nu_p,\, \vec{u}_p)\;\xi_p^*\;(\eta_l + i\gamma_l)\Big\}\Bigg)$$

$$\tilde{c}_2(m,\, k) \;=\; -\frac{m(m-1)}{4} \sum_{\substack{|\nu_p+\nu_l+2\omega_0|+\\+|\vec{v}_p+\vec{v}_l|=0}} \Omega\,(\omega_0 \,+\, \nu_p,\, \vec{u}_p)\;\xi_p\,\xi_l$$

$$\tilde{c}_3(m,\, k) \;=\; \frac{m(m-1)}{4} \sum_{\substack{|\nu_p+\nu_l+3\omega_0|+\\+|\vec{v}_p+\vec{v}_l|=0}} \{\Omega\,(\omega_0 \,+\, \nu_p,\, \vec{u}_p)\;\xi_p\;(\eta_l + i\gamma_l)\;+$$

$$+ \, \Omega \left(2\omega_0 \; + \; \nu_p, \; \vec{u}_p \right) \; (\eta_p + i\gamma_p) \; \xi_l \}$$

$$\tilde{c}_4(m, \, k) \; = \; -\frac{m(m-1)}{4} \sum_{\substack{|\nu_p+\nu_l+4\omega_0|+ \\ +|\vec{v}_p+\vec{v}_l|=0}} \Omega \left(2\omega_0 \; + \; \nu_p, \; \vec{u}_p \right) \; (\eta_p + i\gamma_p) \; (\eta_l + i\gamma_l)$$

$$\tilde{c}_5(m, \, k) \; = \; \frac{m}{4} \left(\sum_{\substack{|\nu_p+\nu_l+2\omega_0|+ \\ +|\vec{v}_p+\vec{v}_l|=0}} \{ k \, \Omega \left(\nu_p, \; \vec{u}_p \right) \; (\eta_p + i\gamma_p) \; (\eta_l + i\gamma_l) \; + \right.$$

$$+ \; (k+1) \, \Omega \left(2\omega_0 \; + \; \nu_p, \; \vec{u}_p \right) \; (\eta_p + i\gamma_p) \; (\eta_l + i\gamma_l) \; -$$

$$- \; (m-1) \, \Omega \left(2\omega_0 \; + \; \nu_p, \; \vec{u}_p \right) \; (\eta_p + i\gamma_p) \; (\eta_l - i\gamma_l) \} \; -$$

$$\left. - \sum_{\substack{|\nu_l-\nu_p+2\omega_0|+ \\ +|\vec{v}_l-\vec{v}_p|=0}} m \, \Omega^* \left(\nu_p, \; \vec{u}_p \right) \; \left(\eta_p^* - i\gamma_p^* \right) \; (\eta_l + i\gamma_l) \right)$$

$$\tilde{c}_6(m, \, k) \; = \; \frac{mk}{2} \sum_{p=-q}^{q} \frac{|\vec{u}_p|^2}{|\vec{u}_p|^4 \; + \; \left(\nu_p \; + \; \omega_0 \right)^2} \; |\xi_p|^2$$

$$\tilde{c}_7(m, \, k) \; = \; -\frac{m+k}{2} \, \alpha \; + \; \frac{mk}{2} \sum_{p=-q}^{q} \frac{|\vec{u}_p|^2}{|\vec{u}_p|^4 \; + \; \nu_p^2} \; |\eta_p \; + \; i\gamma_p|^2 \; +$$

$$+ \; \frac{m^2+k^2}{4} \; \sum_{p=-q}^{q} \frac{|\vec{u}_p|^2}{|\vec{u}_p|^4 \; + \; \nu_p^2} \; \left(|\gamma_p|^2 \; - \; |\eta_p|^2 \right) \; +$$

$$+\,\frac{m+2mk+k}{4}\sum_{p=-q}^{q}\frac{|\vec{u}_p|^2}{|\vec{u}_p|^4\;+\;(\nu_p\;+\;2\,\omega_0)^2}\;|\eta_p\;+\;i\gamma_p|^2\;+$$

$$+\,i\,\frac{m^2-k^2}{4}\sum_{p=-q}^{q}\frac{|\vec{u}_p|^2}{|\vec{u}_p|^4\;+\;\nu_p^2}\;\Big(\eta_p\,\gamma_p^*\;+\;\eta_p^*\gamma_p\Big)\;+$$

$$+\,i\,\frac{m-k}{4}\sum_{p=-q}^{q}\frac{\nu_p\;+\;2\,\omega_0}{|\vec{u}_p|^4\;+\;(\nu_p\;+\;2\,\omega_0)^2}\;|\eta_p\;+\;i\gamma_p|^2$$

Theorem E: *For any initial points* $x_0,\;\; z_0,\;\; \vec{y}_0$ *, for any nonnegative integer* N *and for any positive* L

$$\lim_{\varepsilon\to 0}\;\;\max_{0\le t\le L/\varepsilon^2}\;\;\Big|\Big\langle\tilde{\mathcal{M}}_N^{-1}(\varepsilon^2 t)\,\vec{\mathcal{V}}(I_{m,\,k}(t);\,N)\;-\;\vec{\mathcal{V}}(I_{m,\,k}(0);\,N)\Big\rangle\Big| = 0$$

where the matrix $\tilde{\mathcal{M}}_N(\tau)$ *is the fundamental matrix solution of the system of linear ordinary differential equations with constant coefficients*

$$\frac{d}{d\tau}\,\vec{\mathcal{V}}(a_{m,\,k};\,N)\;=\;\tilde{\mathcal{K}}_N\,\vec{\mathcal{V}}(a_{m,\,k};\,N)$$

generated with the help of the rule (16) in which we use $\tilde{c}_l(m,\,k)$ *instead of* $\breve{c}_l(m,\,k)$*.*

9 Proof of the Theorems

The purpose of this section is to give a detailed proof of the theorems.

9.1 Proof of the Theorem A

1. From the fact that all eigenvalues of the matrix A have negative real parts it follows that there exists a quadratic form $v(\,\vec{y}\,)$ satisfying the conditions

$$C_1\,|\vec{y}|^2\;\le\;v(\,\vec{y}\,)\;\le\;C_2\,|\vec{y}|^2$$

$$A\vec{y}\cdot\mathrm{grad}_{\,\vec{y}}\,v(\,\vec{y}\,)\;\le\;-C_3\,|\vec{y}|^2$$

Here and below C_i are some positive constants the exact values of which are unimportant for us.

2. Let $\hat{L}$ be the generating differential operator of the n-dimensional Markovian diffusion process $\vec{y}$ i.e.

$$\hat{L} \;=\; \frac{\partial}{\partial t} \;+\; A\vec{y}\cdot \operatorname{grad}_{\vec{y}} \;+\; \frac{1}{2}\, BB^{\top} \operatorname{grad}_{\vec{y}} \cdot \operatorname{grad}_{\vec{y}} \tag{17}$$

3. Introduce the function $w(\,\vec{y}\,) \;=\; \exp(\psi\, v(\,\vec{y}\,))$ for wich

$$\operatorname{grad}_{\vec{y}} w \;=\; \psi\, w \cdot \operatorname{grad}_{\vec{y}} v$$

$$\frac{\partial^2 w}{\partial y_m \partial y_k} \;=\; \psi \left(\frac{\partial^2 v}{\partial y_m \partial y_k} \;+\; \psi\, \frac{\partial v}{\partial y_m} \cdot \frac{\partial v}{\partial y_k} \right) w$$

and hence

$$\hat{L}w = \psi \left(A\vec{y}\cdot \operatorname{grad}_{\vec{y}} v \;+\; \frac{1}{2} \sum_{m,\,k\,=\,1}^{n} \left(BB^{\top}\right)_{m,k} \left(\frac{\partial^2 v}{\partial y_m \partial y_k} \;+\; \psi\, \frac{\partial v}{\partial y_m} \cdot \frac{\partial v}{\partial y_k} \right) \right) w$$

From this it follows that there exist constants C_4 and C_5 independent of the value of ψ such that

$$\hat{L}\, w \;\le\; \psi\, \left(-C_3\, |\vec{y}|^2 \;+\; \frac{1}{2} C_4 \;+\; \psi\, C_5\, |\vec{y}|^2 \right) w$$

Taking now $\psi = C_3/2C_5$ we get

$$\hat{L}\, w \;\le\; \frac{\psi}{2} \left(-C_3\, |\vec{y}|^2 \;+\; C_4 \right) w \tag{18}$$

4. Define the constant χ as the maximum of the right side in inequality (18) with respect to the variables $\vec{y}$

$$\chi \;=\; \max_{\vec{y}\in R^n} \left(\frac{\psi}{2} \left(-C_3\, |\vec{y}|^2 + C_4 \right) w \right) \tag{19}$$

Obviously one has

$$0 \;\le\; \chi \;\le\; \frac{\psi}{2}\, C_4\, \exp\left(\psi \frac{C_2\, C_4}{C_3} \right)$$

From (18) and (19) it follows, that the function

$$\hat{w} \;=\; w \,+\, \chi \left(\frac{L}{\varepsilon^2} \,-\, t\right) \tag{20}$$

will satisfy the inequality

$$\hat{L}\,\hat{w} \;\leq\; 0 \tag{21}$$

5. Let $\tilde{s}^{\varepsilon}_t \;=\; s^{\varepsilon}_t \,\wedge\, \frac{L}{\varepsilon^2}$. From (20) and (21) immediately follows that the stochastic process $\hat{w}\,(\,\tilde{s}^{\varepsilon}_t\,)$ is a nonnegative supermartingale, and hence

$$P\left(\tau_\varepsilon \,<\, \frac{L}{\varepsilon^2}\right) \;\leq\; P\left(\sup_{t\,\geq\,0} |\vec{y}\,(\tilde{s}^{\varepsilon}_t)| \geq c(\varepsilon)\right) \leq$$

$$P\left(\sup_{t\geq 0} \hat{w}\,(\tilde{s}^{\varepsilon}_t) \geq \exp(\psi C_1 c^2(\varepsilon))\right) \leq \left(\exp(\psi v(\vec{y}_0)) + \chi \frac{L}{\varepsilon^2}\right) \exp(-\psi C_1 c^2(\varepsilon))$$

The first two inequalities in the sequence shown above are almost obvious, and the last one follows from the property of the stochastic process $\hat{w}\,(\tilde{s}^{\varepsilon}_t)$ to be a nonnegative supermartingale. For finishing the proof take $a = \max(\chi,\; C_2\psi)$ and $b = C_1\psi$.

9.2 Proof of the Theorem B

1. The joint solution of the systems (2), (3) is a Markovian diffusion process in the $(n+2)$-dimensional Euclidean space. Let L be the generating differential operator of this stochastic process. Separating the orders according to ε we can represent L in the form

$$L \;=\; L_0 \,+\, \varepsilon\, L_\varepsilon \,+\, \varepsilon^2\, L_{\varepsilon^2} \tag{22}$$

where the differential operators L_0 , L_ε , L_{ε^2} are defined as follows

$$L_0 = \omega_0 \left(z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}\right) + \hat{L}, \qquad L_\varepsilon = (\xi - \gamma\, z - \eta\, x)\,\frac{\partial}{\partial z}, \qquad L_{\varepsilon^2} = -\alpha\, z\,\frac{\partial}{\partial z}$$

and $\hat{L}$ is the generating differential operator of the n-dimensional Markovian diffusion process $\vec{y}$ given by (17).

2. Now we wish to show that there exist functions $u^{\varepsilon}_{m,\,k}$ satisfying

$$L_0\, u^{\varepsilon}_{m,\,k} \;=\; -L_{\varepsilon}\, I_{m,\,k} \tag{23}$$

Representing the operator L_{ε} in the form

$$L_{\varepsilon} \;=\; \Big(\xi \;-\; (\eta \,+\, i\gamma)\, \exp(i\,\omega_0\, t)\, I_{0,\,1} \;-\; (\eta \,-\, i\gamma)\, \exp(-i\,\omega_0\, t)\, I_{1,\,0}\Big) \cdot \frac{\partial}{\partial z}$$

and calculating [5]

$$\frac{\partial I_{m,\,k}}{\partial z} \;=\; i\frac{m}{2}\, \exp(i\,\omega_0\, t)\, I_{m-1,\,k} \;-\; i\frac{k}{2}\, \exp(-i\,\omega_0\, t)\, I_{m,k-1} \tag{24}$$

and taking into account property **b** we have

$$L_{\varepsilon}\, I_{m,\,k} \;=\; \frac{i}{2}\Big(m\,\xi\, \exp(i\,\omega_0\, t)\, I_{m-1,\,k} \;-\; k\,\xi\, \exp(-i\,\omega_0\, t)\, I_{m,\,k-1} \,+$$

$$[\, k\,(\eta \,+\, i\gamma) \;-\; m\,(\eta \,-\, i\gamma)]\; I_{m,\,k} \,+$$

$$k\,(\eta \,-\, i\gamma)\, \exp(-i\,2\,\omega_0\, t)\, I_{m+1,\,k-1} \;-\; m\,(\eta \,+\, i\gamma)\, \exp(i\,2\,\omega_0\, t)\, I_{m-1,\,k+1}\Big)$$

Looking for the $u^{\varepsilon}_{m,\,k}$ in analogous form

$$u^{\varepsilon}_{m,k} \;=\; \frac{i}{2}\Big(-\,m\,a_1\, \exp(i\,\omega_0\, t)\, I_{m-1,\,k} \;+\; k\,a_1^{*}\, \exp(-i\,\omega_0\, t)\, I_{m,\,k-1} \,+$$

$$(m\,a_2^{*} \;-\; k\,a_2)\; I_{m,\,k} \,+$$

$$m\,a_3\, \exp(i\,2\,\omega_0\, t)\, I_{m-1,\,k+1} \;-\; k\,a_3^{*}\, \exp(-i\,2\,\omega_0\, t)\, I_{m+1,\,k-1}\Big)$$

[5] Starting from this point it is convenient to extend the definition of the function $I_{p,\,q}$ to negative indices assuming that if $p < 0$ or $q < 0$ then $I_{p,\,q} \;\equiv\; 0$. In general, after this extension one has to be careful with respect to the application of the property **b**, but we have not to worry about it, because the only source of lowering indices in this paper is differentiation and hence if the function $I_{p,\,q}$ with negative index will appear we shall have automatically zero multiplyer in front of it.

we get the system defining the unknown a_l

$$\begin{cases} \hat{L}\, a_1 \;+\; i\omega_0\, a_1 \;=\; \xi \\ \\ \hat{L}\, a_2 \;=\; \eta \;+\; i\gamma \\ \\ \hat{L}\, a_3 \;+\; i\, 2\, \omega_0\, a_3 \;=\; \eta \;+\; i\gamma \end{cases} \tag{25}$$

Choosing a_l in the form $a_l \;=\; \vec{a}_l(t) \cdot \vec{y}$, where

$$\vec{a}_l(t) \;=\; \sum_{p=-\infty}^{+\infty} \vec{a}_{l,\,p}\; \exp(i\, \nu_p\, t) \tag{26}$$

and taking into account that

$$\hat{L}\, a_l \;=\; \left(\frac{d\vec{a}_l}{dt} \;+\; A^\top\, \vec{a}_l \right) \cdot \vec{y}$$

we reduce the system (25) to a system of algebraic equations for the Fourier coefficients

$$\begin{cases} \Lambda^\top(\omega_0 \;+\; \nu_p)\, \vec{a}_{1,\,p} \;=\; \vec{h}_p \\ \\ \Lambda^\top(\nu_p)\, \vec{a}_{2,\,p} \;=\; \vec{b}_p \;+\; i\vec{d}_p \\ \\ \Lambda^\top(2\, \omega_0 \;+\; \nu_p)\, \vec{a}_{3,\,p} \;=\; \vec{b}_p \;+\; i\vec{d}_p \end{cases} \tag{27}$$

where we have used the notation

$$\Lambda(\omega) \;=\; A \;+\; i\, \omega\, I.$$

So among the characteristic roots of A we have no purely imaginary or zero values the matrix $\Lambda(\omega)$ is invertiable for an arbitrary real ω and hence the system (27) has a unique solution, which can be expressed as follows

$$\vec{a}_{1,\,p} \;=\; \Lambda^{-\top}(\omega_0 \;+\; \nu_p)\; \vec{h}_p \;=\; \frac{\Lambda^*(\omega_0 \;+\; \nu_p)}{A^\top\, A^\top \;+\; (\omega_0 \;+\; \nu_p)^2 I}\; \vec{h}_p$$

$$\vec{a}_{2,\,p} \;=\; \Lambda^{-\top}(\nu_p)\; \left(\vec{b}_p \;+\; i\vec{d}_p\right) \;=\; \frac{\Lambda^*(\nu_p)}{A^\top\, A^\top \;+\; \nu_p^2 I}\; \left(\vec{b}_p \;+\; i\vec{d}_p\right)$$

$$\vec{a}_{3,\,p} \;=\; \Lambda^{-\top}(2\omega_0 \,+\, \nu_p)\, \left(\vec{b}_p \,+\, i\vec{d}_p\right) \;=\; \frac{\Lambda^{*}(2\omega_0 \,+\, \nu_p)}{A^{\top}\,A^{\top} \,+\, (2\omega_0 \,+\, \nu_p)^2 I}\, \left(\vec{b}_p \,+\, i\vec{d}_p\right)$$

Using the estimate

$$\left|\Lambda^{-\top}(\omega)\right| \;\leq\; \frac{1}{\delta_s^2}$$

which is valid for an arbitrary real ω we get for $\vec{a}_{l,\,p}$

$$|\vec{a}_{l,\,p}| \;\leq\; \frac{1}{\delta_s^2}\, \left(\left|\vec{h}_p\right| + \left|\vec{b}_p\right| + \left|\vec{d}_p\right|\right)$$

which together with (4) guarantees the absolute convergence and the possibility of differentiating the series (26) term by term.

3. Calculating $L_{\varepsilon^2}\, I_{m,\,k}$ and $L_{\varepsilon}\, u^{\varepsilon}_{m,\,k}$ we get

$$\begin{array}{l} L_{\varepsilon^2}\, I_{m,\,k} \;+\; L_{\varepsilon}\, u^{\varepsilon}_{m,\,k} \;= \\ c_2(m,\,k)\, I_{m-2,\,k} \;+\; c_2^{*}(k,\,m)\, I_{m,\,k-2} \;+ \\ c_1(m,\,k)\, I_{m-1,\,k} \;+\; c_1^{*}(k,\,m)\, I_{m,\,k-1} \;+ \\ c_3(m,\,k)\, I_{m-2,\,k+1} \;+\; c_3^{*}(k,\,m)\, I_{m+1,\,k-2} \;+ \\ c_4(m,\,k)\, I_{m-2,\,k+2} \;+\; c_4^{*}(k,\,m)\, I_{m+2,\,k-2} \;+ \\ c_5(m,\,k)\, I_{m-1,\,k+1} \;+\; c_5^{*}(k,\,m)\, I_{m+1,\,k-1} \;+ \\ c_6(m,\,k)\, I_{m-1,\,k-1} \;+\; c_7(m,\,k)\, I_{m,\,k} \end{array} \tag{28}$$

where the functions $c_l(m,\, k)$ are given by the following expressions

$$
\begin{aligned}
c_1(m,\, k) \;&=\; \tfrac{m}{4}\, \Big[\, (\, k\, a_2 \;-\; m\, a_2^* \;+\; (k+1)\, a_3\,)\, \xi \;+ \\
&\quad k\, (a_1 \;+\; a_1^*)(\eta \;+\; i\gamma) \;-\; (m-1)\, a_1\, (\eta \;-\; i\gamma)\Big]\; \exp(i\, \omega_0\, t) \\
c_2(m,\, k) \;&=\; \tfrac{m}{4}\, \Big[\, (m-1)\, a_1\, \xi\Big]\; \exp(i\, 2\, \omega_0\, t) \\
c_3(m,\, k) \;&=\; -\tfrac{m}{4}\, \Big[\, (m-1)\, a_1\, (\eta \;+\; i\gamma) \;+\; (m-1)\, a_3\, \xi\,\Big]\; \exp(i\, 3\, \omega_0\, t) \\
c_4(m,\, k) \;&=\; \tfrac{m}{4}\, \Big[\, (m-1)\, a_3\, (\eta \;+\; i\gamma)\,\Big]\; \exp(i\, 4\, \omega_0\, t) \\
c_5(m,\, k) \;&=\; \tfrac{m}{4}\Big[2\alpha \;-\; (\, k\, a_2 \;-\; m\, a_2^* \;+\; (k+1)\, a_3)\, (\eta \;+\; i\gamma) \;+ \\
&\quad (m-1)\, a_3\, (\eta \;-\; i\gamma)\Big]\; \exp(i\, 2\, \omega_0\, t) \\
c_6(m,\, k) \;&=\; -\tfrac{m}{4}\, \Big[k\, (a_1 \;+\; a_1^*)\, \xi\Big] \\
c_7(m,\, k) \;&=\; -\tfrac{m}{4}\Big[2\alpha \;+\; (k\, a_2 \;-\; m\, a_2^* \;+\; (k+1)\, a_3)(\eta \;-\; i\gamma)\Big] \;- \\
&\quad \tfrac{k}{4}\Big[2\alpha \;+\; (m\, a_2^* \;-\; k\, a_2 \;+\; (m+1)\, a_3^*)(\eta \;+\; i\gamma)\Big]
\end{aligned}
$$

Note that $c_6(m,\, k) \;=\; c_6^*(k,\, m)$ and $c_7(m,\, k) \;=\; c_7^*(k,\, m)$.

4. Introduce a $(n\times n)$ matrix $K(\vec{y}) = \vec{y}\cdot\vec{y}^{\top}$ with the elements $k_{ij} = y_i y_j$. It is easy to check, that this matrix satisfies the equation

$$\hat{L}\, K \;=\; AK \;+\; KA^{\top} \;+\; B\, B^{\top}$$

The usefulness of this matrix for the following is connected with the fact that for arbitrary complex vectors $\vec{a}$ and $\vec{c}$

$$(\vec{a}\cdot\vec{y}\,)\cdot(\vec{c}\cdot\vec{y}\,) \;=\; K\,\vec{a}\cdot\vec{c}^{\,*} \;=\; K\,\vec{c}\cdot\vec{a}^{\,*} \tag{29}$$

5. Define the $(n \times n)$ matrix-function $P_\omega = P_\omega(\,\vec{y}\,)$ with the help of the integral

$$P_\omega \;=\; -\int\limits_0^\infty \exp(i\omega\tau)\exp(A\tau)K(\vec{y})\exp(A^{\top}\tau)\, d\tau \tag{30}$$

This integral converges because all the characteristic roots of A have negative real parts. Introduce the new integration variable $\tau' = \tau + t$, where t is some parameter. Then (30) becomes

$$P_\omega \;=\; -\int\limits_t^\infty \exp(i\omega(\tau'-t))\exp(A(\tau'-t))K(\vec{y})\exp(A^\top(\tau'-t))\,d\tau' \qquad (31)$$

Differentiating (31) with respect to t and using that due to (30) P_ω does not depend on t , we obtain

$$\frac{dP_\omega}{dt} \;=\; K \;-\; AP_\omega \;-\; P_\omega A^\top \;-\; i\omega P_\omega \;=\; 0 \qquad (32)$$

Calculating $\hat{L}\,P_\omega$ and taking into account (32) we get

$$\hat{L}\,P_\omega \;=\; AP_\omega \;+\; P_\omega A^\top \;-\; C(\omega) \;=\; -i\omega P_\omega \;+\; K \;-\; C(\omega) \qquad (33)$$

where we have introduced the notation

$$C(\omega) \;=\; \int\limits_0^\infty \exp(i\omega\tau)\exp(A\tau)BB^\top\exp(A^\top\tau)\,d\tau, \qquad C(0)=D$$

For the following let us rewrite (33) in the form

$$\hat{L}\,P_\omega \;+\; i\omega P_\omega \;=\; K \;-\; C(\omega) \qquad (34)$$

Note that for some positive constant C_1 the norms of the matrices P_ω and $C(\omega)$ can be estimated uniformly with respect to real ω as follows (using 6)

$$|\,P_\omega\,| \;\le\; \frac{C_1}{\delta_s^2}\,|\vec{y}|^2, \qquad |\,C(\omega)\,| \;\le\; \frac{C_1}{\delta_s^2} \qquad (35)$$

6. Now we wish to show that there exist functions $g_l(m,\,k)$ [6] satisfying

$$\hat{L}\,g_l(m,\,k) \;+\; c_l(m,\,k) \;=\; \bar{c}_l(m,\,k), \qquad l \;=\; 1,\ldots,7$$

and these functions have continuous first and continuous first and second derivatives with respect to the variables t and $\vec{y}$ respectively, and these

[6] Of course, $g_l(m,\,k)$ like $c_l(m,\,k)$ are also functions of t and $\vec{y}$

functions together with the above derivative are bounded with respect to the variable t for fixed values of $m,\ k,\ \vec{y}$.

We shall show this for $l = 2$ and the rest can be done by analogy.

Due to (29) and the reality of the vector $\vec{h}$ (that is $\vec{h} = \vec{h}^{\,*}$) we have

$$c_2(m,\ k) \ = \ \frac{m(m-1)}{4}\, a_1\, \xi\, \exp(i\, 2\, \omega_0\, t) \ =$$

$$\frac{m(m-1)}{4} \left(K\, \vec{a}_1 \cdot \vec{h}^{\,*}\right) \exp(i\, 2\, \omega_0\, t) \ = \ \frac{m(m-1)}{4} \left(K\, \vec{a}_1 \cdot \vec{h}\right) \exp(i\, 2\, \omega_0\, t)$$

Substituting in the last expression the Fourier series of the vectors $\vec{a}_1$ and $\vec{h}$ we transform $c_2(m,\ k)$ into the form

$$c_2(m,\ k) \ = \ \frac{m(m-1)}{4} \sum_{p,\, l\, =\, -\infty}^{\infty} \left(K\, \vec{a}_{1,\,p} \cdot \vec{h}_l\right)\ \exp(i(\nu_p - \nu_l + 2\omega_0)t) \ =$$

$$\frac{m(m-1)}{4} \left\{ \sum_{\nu_l - \nu_p\, =\, 2\omega_0} K\vec{a}_{1,\,p} \cdot \vec{h}_l + \sum_{\nu_l - \nu_p\, \neq\, 2\omega_0} \left(K\vec{a}_{1,\,p} \cdot \vec{h}_l\right) \exp(i(\nu_p - \nu_l + 2\omega_0)t) \right\}$$

For $\omega \neq 0$ introduce the matrix

$$Q(\omega) \ = \ \frac{i}{\omega} C(\omega) \ - \ P_\omega$$

and denote $P \ = \ -P_0$. Taking into account (34) we have

$$\hat{L}\ \left(Q(\omega) \exp(i\omega t)\right) \ = \ -\, K\ \exp(i\omega t) \qquad \text{and} \qquad \hat{L}\, P \ = \ D \ - \ K$$

Choosing now

$$g_2(m,\ k) \ = \ \frac{m(m-1)}{4} \left\{ \sum_{\nu_l - \nu_p\, =\, 2\omega_0} P\vec{a}_{1,\,p} \cdot \vec{h}_l + \right.$$

$$\left. \sum_{\nu_l - \nu_p\, \neq\, 2\omega_0} \left(Q(\nu_p - \nu_l + 2\omega_0)\vec{a}_{1,\,p} \cdot \vec{h}_l\right)\ \exp(i(\nu_p - \nu_l + 2\omega_0)t) \right\}$$

we obtain

$$\hat{L}\, g_2(m,\ k)\ +\ c_2(m,\ k)\ =\ \frac{m(m-1)}{4} \sum_{\nu_l - \nu_p\, =\, 2\omega_0} D\, \vec{a}_{1,\,p} \cdot \vec{h}_l$$

which just coincides with the expression for $\bar{c}_2(m,\ k)$ if we take into account that

$$D\, \vec{a}_{1,\,p}\ =\ D\, \frac{\Lambda^*(\omega_0\ +\ \nu_p)}{A^\top\, A^\top\ +\ (\omega_0\ +\ \nu_p)^2 I}\, \vec{h}_p\ =\ -\Psi^*(\omega_0\ +\ \nu_p)\, \vec{h}_p \tag{36}$$

Due to (35) and (5) we have

$$\max\left\{\, |\, P\, |\, ,\ \max_{\nu_l - \nu_p\, \neq\, 2\omega_0} |\, Q(\nu_p - \nu_l + 2\omega_0)\, |\, \right\}\ \leq\ \frac{C_1}{\delta_s^2}\left(|\vec{y}|^2 + \frac{1}{\delta_f^2}\right)$$

and hence, as it can be easily shown, the series defining the function $g_2(m,\, k)$ converges absolutely with

$$|\, g_2(m,\ k)\, |\ \leq\ \frac{m(m-1)}{4}\frac{C_1}{\delta_s^2}\left(|\vec{y}|^2 + \frac{1}{\delta_f^2}\right)\left(\sum_{p=-\infty}^{\infty} |\vec{a}_{1,\,p}|\right)\left(\sum_{p=-\infty}^{\infty} \left|\vec{h}_p\right|\right) \tag{37}$$

The function $g_2(m,\ k)$ is a quadratic polynomial in $\vec{y}$ and so we need to worry about their partial derivative with respect to t only. Expressing

$$\frac{\partial g_2(m,\ k)}{\partial t}\ =\ \frac{m(m-1)}{4} \sum_{\mu_{p,\,l}\, \neq\, 0} i\, \mu_{p,\,l}\, \left(Q(\mu_{p,\,l})\, \vec{a}_{1,\,p} \cdot \vec{h}_l\right)\, \exp(i\mu_{p,\,l} t)$$

where $\mu_{p,\,l}\ =\ \nu_p\ -\ \nu_l\ +\ 2\omega_0$ and using the very rough estimate for $|\mu_{p,\,l}|$

$$|\mu_{p,\,l}|\ \leq\ C_2(1 + |\nu_p|)(1 + |\nu_l|), \qquad C_2\ =\ \max\{\, 1,\ 2|\omega_0|\, \}$$

we get that the series defining the partial derivative with respect to the variable t converges absolutely with

$$\left|\, \frac{\partial g_2(m,\ k)}{\partial t}\, \right|\ \leq\ \frac{m(m-1)}{4}\frac{C_1}{\delta_s^2}\left(|\vec{y}|^2 + \frac{1}{\delta_f^2}\right) C_2 \cdot$$

$$\cdot \left(\sum_{p=-\infty}^{\infty} (1 + |\nu_p|)\, |\vec{a}_{1,\,p}|\right)\left(\sum_{p=-\infty}^{\infty} (1 + |\nu_p|)\left|\vec{h}_p\right|\right) \tag{38}$$

Note that expressions similar to (36) hold also for $D\vec{a}_{2,\,p}$ and $D\vec{a}_{3,\,p}$

$$D\,\vec{a}_{2,\,p} \;=\; -\Psi^*(\nu_p)\;\left(\vec{b}_p \;+\; i\vec{d}_p\right)$$

$$D\,\vec{a}_{3,\,p} \;=\; -\Psi^*(2\omega_0 \;+\; \nu_p)\;\left(\vec{b}_p \;+\; i\vec{d}_p\right)$$

7. Defining $u^{\varepsilon^2}_{m,\,k}$ as

$$\begin{aligned} u^{\varepsilon^2}_{m,\,k} \;=\;\; & g_2(m,\;k)\,I_{m-2,\,k} \;+\; g_2^*(k,\;m)\,I_{m,\,k-2} \;+ \\ & g_1(m,\;k)\,I_{m-1,\,k} \;+\; g_1^*(k,\;m)\,I_{m,\,k-1} \;+ \\ & g_3(m,\;k)\,I_{m-2,\,k+1} \;+\; g_3^*(k,\;m)\,I_{m+1,\,k-2} \;+ \\ & g_4(m,\;k)\,I_{m-2,\,k+2} \;+\; g_4^*(k,\;m)\,I_{m+2,\,k-2} \;+ \\ & g_5(m,\;k)\,I_{m-1,\,k+1} \;+\; g_5^*(k,\;m)\,I_{m+1,\,k-1} \;+ \\ & g_6(m,\;k)\,I_{m-1,\,k-1} \;+\; g_7(m,\;k)\,I_{m,\,k} \end{aligned}$$

and acting on the function

$$\tilde{I}_{m,\,k} = I_{m,\,k} \;+\; \varepsilon\;u^{\varepsilon}_{m,\,k} \;+\; \varepsilon^2\;u^{\varepsilon^2}_{m,\,k} \tag{39}$$

by means of the operator L we have

$$\begin{aligned} L\;\tilde{I}_{m,\,k} \;=\; & \left(L_0 \;+\; \varepsilon L_\varepsilon \;+\; \varepsilon^2 L_{\varepsilon^2}\right)\;\tilde{I}_{m,\,k} \;= \\ & L_0\;I_{m,\,k} \;+\; \varepsilon\left(L_0\;u^{\varepsilon}_{m,\,k} \;+\; L_\varepsilon\;I_{m,\,k}\right) \;+ \\ & \varepsilon^2\left(L_0\;u^{\varepsilon^2}_{m,\,k} \;+\; L_{\varepsilon^2}\;I_{m,\,k} \;+\; L_\varepsilon\;u^{\varepsilon}_{m,\,k}\right) \;+\; \varepsilon^3 R_{m,\,k} \end{aligned} \tag{40}$$

where for the remainder $R_{m,\,k}$ we have the expression

$$R_{m,\,k} \;=\; L_{\varepsilon^2}\;u^{\varepsilon}_{m,\,k} \;+\; L_\varepsilon\;u^{\varepsilon^2}_{m,\,k} + \varepsilon L_{\varepsilon^2}\;u^{\varepsilon^2}_{m,\,k} \tag{41}$$

Due to our construction of the functions $I_{m,\,k}$, $u^{\varepsilon}_{m,\,k}$ and $u^{\varepsilon^2}_{m,\,k}$ from (40) it follows that

$$\begin{aligned} L\,\tilde{I}_{m,\,k} \;=\;& \varepsilon^2\,(\bar{c}_2(m,\;k)\,I_{m-2,\,k} + \bar{c}_2^{\,*}(k,\;m)\,I_{m,\,k-2} + \\ & \bar{c}_1(m,\;k)\,I_{m-1,\,k} + \bar{c}_1^{\,*}(k,\;m)\,I_{m,\,k-1} + \\ & \bar{c}_3(m,\;k)\,I_{m-2,\,k+1} + \bar{c}_3^{\,*}(k,\;m)\,I_{m+1,\,k-2} + \\ & \bar{c}_4(m,\;k)\,I_{m-2,\,k+2} + \bar{c}_4^{\,*}(k,\;m)\,I_{m+2,\,k-2} + \\ & \bar{c}_5(m,\;k)\,I_{m-1,\,k+1} + \bar{c}_5^{\,*}(k,\;m)\,I_{m+1,\,k-1} + \\ & \bar{c}_6(m,\;k)\,I_{m-1,\,k-1} + \bar{c}_7(m,\;k)\,I_{m,\,k}) + \\ & \varepsilon^3\,R_{m,\,k} \end{aligned} \tag{42}$$

8. So

$$|\,I_{p,\,q}\,| \;=\; \left(\frac{r}{2}\right)^{\frac{p+q}{2}} \;\le\; 1 \;+\; \left(\frac{r}{2}\right)^{\frac{m+k}{2}} \;=\; 1 \;+\; |\,I_{m,\,k}\,|$$

for $p+q \;\le\; m+k$, then for some positive constant C_3 independent of m and k the functions $u^{\varepsilon}_{m,\,k}$, $u^{\varepsilon^2}_{m,\,k}$ defined above and $R_{m,\,k}$ can be roughly estimated as follows

$$\left|\,u^{\varepsilon}_{m,\,k}\,\right| \;\le\; C_3(m+k)\,|\vec{y}|\,(1+|\,I_{m,\,k}\,|) \tag{43}$$

$$\left|\,u^{\varepsilon^2}_{m,\,k}\,\right| \;\le\; C_3(m+k)^2\left(1+|\vec{y}|^2\right)(1+|\,I_{m,\,k}\,|) \tag{44}$$

$$|\,R_{m,\,k}\,| \;\le\; C_3(m+k)^3\left(1+|\vec{y}|^3\right)(1+|\,I_{m,\,k}\,|) \tag{45}$$

9. So $|\,I_{m,\,m}\,| \;=\; I_{m,\,m}$ then for some positive constant C_4 independent of m we have from (42)

$$L\,\tilde{I}_{m,\,m} \;\le\; \varepsilon^2\,C_4\,m^2\;(1\;+\;I_{m,\,m})\;+\;\varepsilon^3\,|\,R_{m,\,m}\,| \tag{46}$$

Using the estimate (45) we can rewrite (46) in the form

$$L\,\tilde{I}_{m,\,m} \;\le\; \varepsilon^2\,\mathcal{H}_m\;(1\;+\;I_{m,\,m}) \tag{47}$$

and for $u^{\varepsilon}_{m,\,m} \;+\; \varepsilon\, u^{\varepsilon^2}_{m,\,m}$ we have from (43) and (44)

$$\Big|\; u^{\varepsilon}_{m,\,m} \;+\; \varepsilon\, u^{\varepsilon^2}_{m,\,m} \;\Big| \;\leq\; \mathcal{G}_m \;(1 \;+\; I_{m,\,m}) \tag{48}$$

Here

$$\mathcal{H}_m \;=\; m^2 \;\Big(C_4 \;+\; \varepsilon\, 8\, m\, C_3 \Big(1 \;+\; |\vec{y}|^3\Big)\Big) \tag{49}$$

$$\mathcal{G}_m \;=\; 2\, m\, C_3 \;\Big(|\vec{y}| \;+\; \varepsilon\, 2\, m \Big(1 \;+\; |\vec{y}|^2\Big)\Big) \tag{50}$$

Let $\bar{\mathcal{H}}_m$ and $\bar{\mathcal{G}}_m$ be some positive constants. Consider the function

$$v_m \;=\; \Big(1 \;+\; \tilde{I}_{m,\,m}\Big) \exp\left(-\varepsilon^2 \frac{\bar{\mathcal{H}}_m}{1-\varepsilon\bar{\mathcal{G}}_m}\, t\right)$$

for which after some straightforward calculations we have

$$L\, v_m \;\leq\; \varepsilon^2 \left(\Big(\mathcal{H}_m \;-\; \bar{\mathcal{H}}_m\Big) \;+\; \varepsilon \frac{\bar{\mathcal{H}}_m}{1-\varepsilon\bar{\mathcal{G}}_m} \Big(\mathcal{G}_m \;-\; \bar{\mathcal{C}}_m\Big)\right) \cdot$$

$$\cdot\, (1 \;+\; I_{m,\,m}) \exp\left(-\varepsilon^2 \frac{\bar{\mathcal{H}}_m}{1-\varepsilon\bar{\mathcal{G}}_m}\, t\right) \tag{51}$$

Choosing now

$$\bar{\mathcal{H}}_m \;=\; m^2 \;\Big(C_4 \;+\; \varepsilon\, 8\, m\, C_3 \Big(1 \;+\; c^3(\varepsilon)\Big)\Big)$$

$$\bar{\mathcal{G}}_m \;=\; 2\, m\, C_3 \;\Big(c(\varepsilon) \;+\; \varepsilon\, 2\, m \Big(1 \;+\; c^2(\varepsilon)\Big)\Big)$$

and assuming that ε is small enough to guarantee $1-\varepsilon\bar{\mathcal{G}}_m \;>\; 0$ we get from (51) that $L\, v_m \;\leq\; 0$ on the set $|\vec{y}| \;\leq\; c(\varepsilon)$ and hence

$$\langle v_m\,(s^{\varepsilon}_t)\rangle \;\leq\; \langle v_m\,(0)\rangle \tag{52}$$

Using that with probability one

$$1 \;+\; \tilde{I}_{m,\,m}\,(s^{\varepsilon}_t) \;\leq\; \Big(1+\varepsilon\bar{\mathcal{G}}_m\Big)\,(1 \;+\; I_{m,\,m}\,(s^{\varepsilon}_t))$$

$$1 \; + \; \tilde{I}_{m,\,m}\,(s_t^{\varepsilon}) \;\geq\; \left(1-\varepsilon\bar{\mathcal{G}}_m\right)(1 \; + \; I_{m,\,m}\,(s_t^{\varepsilon}))$$

and hence with probability one

$$\left(1-\varepsilon\bar{\mathcal{G}}_m\right)\;(1 \; + \; I_{m,\,m}\,(s_t^{\varepsilon}))\exp\left(-\varepsilon^2\frac{\bar{\mathcal{H}}_m}{1-\varepsilon\bar{\mathcal{G}}_m}\,t\right) \;\leq\; v_m\,(s_t^{\varepsilon})$$

$$v_m\,(0) \;\leq\; \left(1+\varepsilon\bar{\mathcal{G}}_m\right)\;(1 \; + \; I_{m,\,m}\,(0))$$

we have from (52) the estimate

$$\langle 1 \; + \; I_{m,\,m}\,(s_t^{\varepsilon})\rangle \;\leq\; \frac{1+\varepsilon\bar{\mathcal{G}}_m}{1-\varepsilon\bar{\mathcal{G}}_m}\;\langle 1 \; + \; I_{m,\,m}\,(0)\rangle\;\exp\left(\varepsilon^2\frac{\bar{\mathcal{H}}_m}{1-\varepsilon\bar{\mathcal{G}}_m}\,t\right)$$

which for the following is rewritten in the form

$$\max_{0\leq t\leq L/\varepsilon^2}\;\langle 1 \; + \; I_{m,\,m}\,(s_t^{\varepsilon})\rangle \;\leq\; \bar{\mathcal{D}}_m\;\langle 1 \; + \; I_{m,\,m}\,(0)\rangle \tag{53}$$

where

$$\bar{\mathcal{D}}_m \;=\; \frac{1+\varepsilon\bar{\mathcal{G}}_m}{1-\varepsilon\bar{\mathcal{G}}_m}\exp\left(\frac{L\bar{\mathcal{H}}_m}{1-\varepsilon\bar{\mathcal{G}}_m}\right), \qquad \lim_{\varepsilon\to 0}\bar{\mathcal{D}}_m \;=\; \exp\left(m^2 L C_4\right)$$

10. Denoting $u_{m,\,k} \;=\; u^{\varepsilon}_{m,\,k} + \varepsilon u^{\varepsilon^2}_{m,\,k}$ and introducing the new remainder

$$\begin{aligned}
\tilde{R}_{m,\,k} = R_{m,\,k} \;\; - \;\; & \bar{c}_2(m,\;k)\,u_{m-2,\,k} && - \;\; \bar{c}_2^{\,*}(k,\;m)\,u_{m,\,k-2} && - \\
& \bar{c}_1(m,\;k)\,u_{m-1,\,k} && - \;\; \bar{c}_1^{\,*}(k,\;m)\,u_{m,\,k-1} && - \\
& \bar{c}_3(m,\;k)\,u_{m-2,\,k+1} && - \;\; \bar{c}_3^{\,*}(k,\;m)\,u_{m+1,\,k-2} && - \\
& \bar{c}_4(m,\;k)\,u_{m-2,\,k+2} && - \;\; \bar{c}_4^{\,*}(k,\;m)\,u_{m+2,\,k-2} && - \\
& \bar{c}_5(m,\;k)\,u_{m-1,\,k+1} && - \;\; \bar{c}_5^{\,*}(k,\;m)\,u_{m+1,\,k-1} && - \\
& \bar{c}_6(m,\;k)\,u_{m-1,\,k-1} && - \;\; \bar{c}_7(m,\;k)\,u_{m,\,k}
\end{aligned}$$

which admits the estimate (as follows from (43)-(44))

$$\left|\;\tilde{R}_{m,\,k}\;\right| \;\leq\; C_5(m+k)^3(1+\varepsilon(m+k))\left(1+|\vec{y}|^3\right)(1+|\;I_{m,\,k}\;|) \tag{54}$$

we get from (42)

$$
\begin{aligned}
L\,\tilde I_{m,\,k} \;=\; \varepsilon^2 \Big(& \bar c_2(m,\;k)\,\tilde I_{m-2,\,k} \;+\; \bar c_2^{\,*}(k,\;m)\,\tilde I_{m,\,k-2} \;+\\
& \bar c_1(m,\;k)\,\tilde I_{m-1,\,k} \;+\; \bar c_1^{\,*}(k,\;m)\,\tilde I_{m,\,k-1} \;+\\
& \bar c_3(m,\;k)\,\tilde I_{m-2,\,k+1} \;+\; \bar c_3^{\,*}(k,\;m)\,\tilde I_{m+1,\,k-2} \;+\\
& \bar c_4(m,\;k)\,\tilde I_{m-2,\,k+2} \;+\; \bar c_4^{\,*}(k,\;m)\,\tilde I_{m+2,\,k-2} \;+\\
& \bar c_5(m,\;k)\,\tilde I_{m-1,\,k+1} \;+\; \bar c_5^{\,*}(k,\;m)\,\tilde I_{m+1,\,k-1} \;+\\
& \bar c_6(m,\;k)\,\tilde I_{m-1,\,k-1} \;+\; \bar c_7(m,\;k)\,\tilde I_{m,\,k}\Big) \;+\\
& \varepsilon^3\,\tilde R_{m,\,k}
\end{aligned}
\tag{55}
$$

Let now N be as in the theorem B. Using notation $\vec{\mathcal V}(\,*\,;\,N)$ we can rewrite (55) in the form of the following system

$$
L\,\vec{\mathcal V}(\tilde I_{m,\,k};\,N) \;=\; \varepsilon^2\,\bar{\mathcal K}_N\,\vec{\mathcal V}(\tilde I_{m,\,k};\,N) \;+\; \varepsilon^3\,\vec{\mathcal V}(\tilde R_{m,\,k};\,N) \tag{56}
$$

where the matrix $\bar{\mathcal K}_N$ is the same as in (10).

The matrix $\bar{\mathcal M}_N(\tau)$ is assumed to be the fundamental matrix solution of (10). That means that the matrix $\bar{\mathcal M}_N^{-1}(\varepsilon^2 t)$ satisfies

$$
\frac{d}{dt}\bar{\mathcal M}_N^{-1}(\varepsilon^2 t) \;=\; -\varepsilon^2\,\bar{\mathcal M}_N^{-1}(\varepsilon^2 t)\,\bar{\mathcal K}_N, \qquad \bar{\mathcal M}_N^{-1}(0) \;=\; I \tag{57}
$$

Applying the operator L to the vector $\bar{\mathcal M}_N^{-1}(\varepsilon^2 t)\,\vec{\mathcal V}(\tilde I_{m,\,k}(t);\,N)$ and taking into account (56), (57) we obtain

$$
L\,\Big(\bar{\mathcal M}_N^{-1}(\varepsilon^2 t)\,\vec{\mathcal V}(\tilde I_{m,\,k}(t);\,N)\Big) \;=\; \varepsilon^3\,\bar{\mathcal M}_N^{-1}(\varepsilon^2 t)\,\vec{\mathcal V}(\tilde R_{m,\,k}(t);\,N) \tag{58}
$$

From (58) and Dynkin's formula (see, for example [4]) it follows that

$$
\Big\langle \bar{\mathcal M}_N^{-1}(\varepsilon^2 s_t^\varepsilon)\,\vec{\mathcal V}(\tilde I_{m,\,k}(s_t^\varepsilon);\,N) \;-\; \vec{\mathcal V}(\tilde I_{m,\,k}(0);\,N)\Big\rangle \;=
$$

$$
=\; \varepsilon^3\,\left\langle \int\limits_0^{s_t^\varepsilon} \bar{\mathcal M}_N^{-1}(\varepsilon^2\tau)\,\vec{\mathcal V}(\tilde R_{m,\,k}(\tau);\,N)\,d\tau \right\rangle
$$

or, equivalently

$$\left\langle \bar{\mathcal{M}}_N^{-1}(\varepsilon^2 s_t^\varepsilon)\,\vec{\mathcal{V}}(I_{m,\,k}(s_t^\varepsilon);\,N)\;-\;\vec{\mathcal{V}}(I_{m,\,k}(0);\,N)\right\rangle\;=$$

$$=\;\varepsilon\cdot\left\langle\vec{\mathcal{V}}(u_{m,\,k}(0);\,N)\;-\;\bar{\mathcal{M}}_N^{-1}(\varepsilon^2 s_t^\varepsilon)\,\vec{\mathcal{V}}(u_{m,\,k}(s_t^\varepsilon);\,N)\right\rangle\;+$$

$$+\;\varepsilon^3\left\langle\int_0^{s_t^\varepsilon}\bar{\mathcal{M}}_N^{-1}(\varepsilon^2\tau)\,\vec{\mathcal{V}}(\tilde{R}_{m,\,k}(\tau);\,N)\,d\tau\right\rangle \tag{59}$$

From (59) we obtain

$$\max_{0\le t\le L/\varepsilon^2}\;\left|\left\langle\bar{\mathcal{M}}_N^{-1}(\varepsilon^2 s_t^\varepsilon)\,\vec{\mathcal{V}}(I_{m,\,k}(s_t^\varepsilon);\,N)\;-\;\vec{\mathcal{V}}(I_{m,\,k}(0);\,N)\right\rangle\right|\;\le$$

$$\le\;\varepsilon\cdot\max_{0\le\tau\le L}\left|\bar{\mathcal{M}}_N^{-1}(\tau)\right|\max_{0\le t\le L/\varepsilon^2}\left\langle 2\left|\vec{\mathcal{V}}(u_{m,\,k}(s_t^\varepsilon);\,N)\right|\;+\;L\left|\vec{\mathcal{V}}(\tilde{R}_{m,\,k}(s_t^\varepsilon);\,N)\right|\right\rangle \tag{60}$$

Let us define now $[m]$ as the smallest integer which is bigger or equal to m. Using (43), (44), (54) and simple inequalities like

$$|\,\vec{y}\,|\;\le\;1\;+\;|\,\vec{y}\,|^2$$

$$1\;+\;|\,I_{m,\,k}\,|\;\le\;2\left(1\;+\;I_{[\frac{N}{2}],\,[\frac{N}{2}]}\right),\qquad m\;+\;k\;\le\;N$$

we can obtain

$$\max\left\{\left|\vec{\mathcal{V}}(u_{m,\,k};\,N)\right|,\;\left|\vec{\mathcal{V}}(\tilde{R}_{m,\,k};\,N)\right|\right\}\;\le$$

$$\le\;C_6N^5\,(1+\varepsilon N)\left(1+|\vec{y}|^3\right)\left(1+I_{[\frac{N}{2}],\,[\frac{N}{2}]}\right) \tag{61}$$

Taking into account (53) we have from (61)

$$\max_{0\le t\le L/\varepsilon^2}\left\langle 2\left|\vec{\mathcal{V}}(u_{m,\,k}(s_t^\varepsilon);\,N)\right|\;+\;L\left|\vec{\mathcal{V}}(\tilde{R}_{m,\,k}(s_t^\varepsilon);\,N)\right|\right\rangle\;\le$$

$$\le\;C_6N^5\,(1+\varepsilon N)\left(1+c^3(\varepsilon)\right)(2+L)\bar{\mathcal{D}}_{[\frac{N}{2}]}\left(1+I_{[\frac{N}{2}],\,[\frac{N}{2}]}(0)\right)$$

which together with

$$\max_{0\le\tau\le L}\left|\bar{\mathcal{M}}_N^{-1}(\tau)\right| \;\le\; \exp(C_7 N^2)$$

and (60) gives the final estimate we need

$$\max_{0\le t\le L/\varepsilon^2}\quad\left|\left\langle \bar{\mathcal{M}}_N^{-1}(\varepsilon^2 s_t^\varepsilon)\,\vec{\mathcal{V}}(I_{m,\,k}(s_t^\varepsilon);\,N)\;-\;\vec{\mathcal{V}}(I_{m,\,k}(0);\,N)\right\rangle\right| \;\le$$

$$\le\;\varepsilon\,\left(1+c^3(\varepsilon)\right)\,P\,\left(1+I_{[\frac{N}{2}],\,[\frac{N}{2}]}(0)\right) \tag{62}$$

where

$$P \;=\; C_6 N^5\,(1+\varepsilon N)\,(2+L)\bar{\mathcal{D}}_{[\frac{N}{2}]}\exp(C_7 N^2)$$

Taking the limit $\varepsilon\;\to\;0$ we have from (62) the proof of the theorem B with speed of convergence $\varepsilon\, c^3(\varepsilon)$.

11. To prove the remark to the theorem B we apply the operator L to the function $\vec{a}_N(\varepsilon^2 t)\cdot\vec{\mathcal{V}}(\tilde{I}_{m,\,k}(t);\,N)$ with the result that

$$L\,\left(\vec{a}_N(\varepsilon^2 t)\cdot\vec{\mathcal{V}}(\tilde{I}_{m,\,k}(t);\,N)\right)\;=\;\varepsilon^3\,\vec{a}_N(\varepsilon^2 t)\cdot\vec{\mathcal{V}}(\tilde{R}_{m,\,k}(t);\,N) \tag{63}$$

The rest of the proof is just repeating all the steps from the previous point with the usage (63) instead of (58).

9.3 Sketch of the Proof of the Theorem D

1. The solution of the system (14) is a Markovian diffusion process in the 2-dimensional Euclidean space. Let L be a generating differential operator of this stochastic process. Separating the orders according to ε we can represent L in the form

$$L\;=\;L_0\;+\;\varepsilon^2\,L_{\varepsilon^2} \tag{64}$$

where the differential operators L_0 and L_{ε^2} are defined as follows

$$L_0\;=\;\frac{\partial}{\partial t}\;+\;\omega_0\,\left(z\,\frac{\partial}{\partial x}-x\,\frac{\partial}{\partial z}\right)$$

$$L_{\varepsilon^2} = \Big(\, z \, \Big(\Phi \, \vec{d} \cdot \vec{d} \, - \, \alpha \Big) \, + \, x \, \Phi \, \vec{b} \cdot \vec{d} \, - \, \Phi \, \vec{h} \cdot \vec{d} \Big) \, \frac{\partial}{\partial z} +$$

$$+ \, \Big(\, \Phi \, \vec{h} \cdot \vec{h} \, - \, 2x \, \Phi \, \vec{h} \cdot \vec{b} \, - \, 2z \, \Phi \, \vec{h} \cdot \vec{d} \, +$$

$$+ \, x^2 \, \Phi \, \vec{b} \cdot \vec{b} \, + \, 2xz \, \Phi \, \vec{d} \cdot \vec{b} \, + \, z^2 \, \Phi \, \vec{d} \cdot \vec{d} \Big) \, \frac{\partial^2}{\partial z^2}$$

2. Now we wish to calculate $L_{\varepsilon^2} \, I_{m,\,k}$. Representing the operator L_{ε^2} in the form

$$L_{\varepsilon^2} \; = \; \Big(- \, \Phi \vec{h} \cdot \vec{d} \, + \, \Big[\, \Phi \vec{b} \cdot \vec{d} \, - \, i \, \alpha \, + \, i \, \Phi \vec{d} \cdot \vec{d} \Big] \exp(i \, \omega_0 \, t) \, I_{0,\,1} \, +$$

$$+ \, \Big[\, \Phi \vec{b} \cdot \vec{d} \, + \, i \, \alpha \, - \, i \, \Phi \vec{d} \cdot \vec{d} \Big] \exp(-i \, \omega_0 \, t) \, I_{1,\,0} \Big) \cdot \frac{\partial}{\partial z} \, +$$

$$+ \, \Big(\Phi \vec{h} \cdot \vec{h} \, - \, 2 \, \Big[\, \Phi \vec{h} \cdot \vec{b} \, + \, i \, \Phi \vec{h} \cdot \vec{d} \Big] \exp(i \, \omega_0 \, t) \, I_{0,\,1} \, -$$

$$- \, 2 \, \Big[\, \Phi \vec{h} \cdot \vec{b} \, - \, i \, \Phi \vec{h} \cdot \vec{d} \Big] \exp(-i \, \omega_0 \, t) \, I_{1,\,0} \, + \, 2 \, \Big[\, \Phi \vec{b} \cdot \vec{b} \, + \, \Phi \vec{b} \cdot \vec{d} \Big] \; I_{1,\,1} \, +$$

$$+ \, \Big[\, \Phi \vec{b} \cdot \vec{b} \, - \, \Phi \vec{d} \cdot \vec{d} \, + \, 2 \, i \, \Phi \vec{d} \cdot \vec{b} \Big] \exp(i \, 2 \, \omega_0 \, t) \, I_{0,\,2} \, +$$

$$+ \, \Big[\, \Phi \vec{b} \cdot \vec{b} \, - \, \Phi \vec{d} \cdot \vec{d} \, - \, 2 \, i \, \Phi \vec{d} \cdot \vec{b} \Big] \exp(-i \, 2 \, \omega_0 \, t) \, I_{2,\,0} \Big) \cdot \frac{\partial^2}{\partial z^2}$$

taking into account (24), property **b** and the expression

$$\frac{\partial^2 I_{m,\,k}}{\partial z^2} \; = \; \frac{mk}{2} \, I_{m-1,\,k-1} \, -$$

$$- \, \frac{m(m-1)}{4} \; \exp(i \, 2 \, \omega_0 \, t) \; I_{m-2,\,k} \; - \; \frac{k(k-1)}{4} \; \exp(-i \, 2 \, \omega_0 \, t) \; I_{m,k-2}$$

we obtain that $L_{\varepsilon^2} I_{m,k}$ is given by the right hand side of (28) with $c_l(m, k)$ as follows

$$
\begin{aligned}
c_1(m,\ k) &= \tfrac{m}{2}\left[(m-2k-1)\,\Phi\,\vec{h}\cdot\vec{b} \,-\, i\,(m+2k)\,\Phi\,\vec{h}\cdot\vec{d}\right]\,\exp(i\,\omega_0\,t)\\
c_2(m,\ k) &= -\tfrac{m(m-1)}{4}\,\left[\Phi\,\vec{h}\cdot\vec{h}\right]\,\exp(i\,2\,\omega_0\,t)\\
c_3(m,\ k) &= \tfrac{m(m-1)}{2}\,\left[\Phi\,\vec{h}\cdot\left(\vec{b}\,-\,i\,\vec{d}\right)\right]\,\exp(i\,3\,\omega_0\,t)\\
c_4(m,\ k) &= -\tfrac{m(m-1)}{4}\,\left[\Phi\,\left(\vec{b}\,+\,i\,\vec{d}\right)\cdot\left(\vec{b}\,-\,i\,\vec{d}\right)\right]\,\exp(i\,4\,\omega_0\,t)\\
c_5(m,\ k) &= \tfrac{m}{2}\Big[\alpha\,-\,(m+k)\,\Phi\,\vec{d}\cdot\vec{d}\,+\,(k-m+1)\,\Phi\,\vec{b}\cdot\vec{b}\,+\\
&\qquad i\,(2k+1)\,\Phi\,\vec{b}\cdot\vec{d}\Big]\,\exp(i\,2\,\omega_0\,t)\\
c_6(m,\ k) &= \tfrac{mk}{2}\,\Phi\,\vec{h}\cdot\vec{h}\\
c_7(m,\ k) &= -\tfrac{m+k}{2}\,\alpha\,+\,\tfrac{4mk-m(m-1)-k(k-1)}{4}\,\Phi\,\vec{b}\cdot\vec{b}\,+\\
&\qquad \tfrac{4mk+m(m+1)+k(k+1)}{4}\,\Phi\,\vec{d}\cdot\vec{d}\,+\,i\,\tfrac{m^2-k^2}{2}\,\Phi\,\vec{d}\cdot\vec{b}
\end{aligned}
$$

3. Now, like in the proof of theorem B, we want to show that there exist for fixed m and k functions $g_l(m, k)$ bounded in t and satisfying

$$\frac{\partial g_l(m,\ k)}{\partial t}\,+\,c_l(m,\ k)\,=\,\check{c}_l(m,\ k),\qquad l\,=\,1,\ldots,7$$

We shall show it for $l = 4$ and the rest can be done by analogy.

Substituting in the expression for $c_4(m,k)$ the Fourier series of the vectors $\vec{b}$ and $\vec{d}$ we obtain

$$c_4(m,\ k)\,=\,-\frac{m(m-1)}{2}\sum_{p,\,l\,=\,-\infty}^{\infty}\left[\Phi\,\left(\vec{b}_p\,+\,\vec{d}_p\right)\cdot\left(\vec{b}_l\,-\,\vec{d}_l\right)\right]\,\exp(i\,\mu_{p,\,l}t)\,=$$

$$=\,\check{c}_4(m,\ k)\,-\,\frac{m(m-1)}{4}\sum_{\mu_{p,\,l}\,\neq\,0}\left[\Phi\,\left(\vec{b}_p\,+\,\vec{d}_p\right)\cdot\left(\vec{b}_l\,-\,\vec{d}_l\right)\right]\,\exp(i\,\mu_{p,\,l}t)$$

where $\mu_{p,\,l}\,=\,\nu_p-\nu_l+4\omega_0\,.$

So the problem will be solved if the series

$$c_4(m,\ k) \ = \ i\,\frac{m(m-1)}{4} \sum_{\mu_{p,\,l} \neq 0} \frac{\Phi\,\left(\vec{b}_p \,+\, \vec{d}_p\right) \cdot \left(\vec{b}_l \,-\, \vec{d}_l\right)}{\mu_{p,\,l}} \ \exp(i\,\mu_{p,\,l} t)$$

converges and can be differentiated term by term. The absolute convergence and differentiability can be easily shown using (4) and (5) with the final estimates

$$|\,g_4(m,\ k)\,| \ \leq \ \frac{m(m-1)}{2}\frac{|\Phi|}{\delta_f^2}\left(\sum_{p=-\infty}^{\infty} \left|\vec{b}_p\right| \,+\, \left|\vec{d}_p\right|\right)^2$$

$$\left|\,\frac{\partial g_4(m,\ k)}{\partial t}\,\right| \ \leq \ \frac{m(m-1)}{2}\ |\Phi| \left(\sum_{p=-\infty}^{\infty} \left|\vec{b}_p\right| \,+\, \left|\vec{d}_p\right|\right)^2$$

4. The rest follows the simplified version of the proof of the theorem B.

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